PAIR ANNIHILATION WITH AN ELECTRON AND A POSITRON

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Post 35.

Here's a slightly more involved example of conservation of energy and momentum in relativity. Suppose an electron with momentum p_e hits a positron (anti-electron) and they annihilate each other (a process known as *pair annihilation*), producing two photons.

In passing, it's worth noting that pair annihilation must produce at least 2 photons. If we looked at the problem in the centre of momentum frame, the total momentum of the system must be zero both before and after the collision. If only one photon were produced, it is impossible for it to have zero momentum.

Before the collision, we have

$$E_{tot}=mc^2+\sqrt{mc^2+p_e^2c^2} \tag{1}$$

After the collision, suppose that one photon emerges at $60^{\circ}=\frac{\pi}{3}$ radians to the incident electron's direction. From conservation of momentum, we have

$$\begin{aligned} p_1\sin\frac{\pi}{3} &= p_2\sin\theta && (2)\\ p_1\cos\frac{\pi}{3}+p_2\cos\theta &= p_e && (3)\end{aligned}$$

where θ is the angle at which photon 2 emerges. We can eliminate θ as follows (using $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$, $\cos\frac{\pi}{3}=\frac{1}{2}$):

$$\begin{aligned} p_2^2\left(\sin^2\theta+\cos^2\theta\right) &= \frac{3}{4}p_1^2+\left(p_e-\frac{1}{2}p_1\right)^2 && (4)\\ p_2^2 &= p_1^2-p_1p_e+p_e^2 && (5)\end{aligned}$$

For a photon, $E=cp$, so the total energy after the collision is

$$E_{tot}=cp_1+cp_2 \tag{6}$$

From conservation of energy and 1, we get

$$
\begin{aligned}
c^2 p_2^2 &= (E_{tot} - cp_1)^2 && (7)\\
&= \left(mc^2 + \sqrt{m^2c^4 + p_e^2c^2} - cp_1\right)^2 && (8)\\
&= \left(mc^2 + \sqrt{m^2c^4 + p_e^2c^2}\right)^2 - 2\left(mc^2 + \sqrt{m^2c^4 + p_e^2c^2}\right)cp_1 + c^2p_1^2 && (9)
\end{aligned}
$$

We can now use 5:

$$
\begin{aligned}
c^2 p_2^2 &= c^2p_1^2 - c^2p_1p_e + c^2p_e^2 && (10)\\
&= \left(mc^2 + \sqrt{m^2c^4 + p_e^2c^2}\right)^2 - 2\left(mc^2 + \sqrt{m^2c^4 + p_e^2c^2}\right)cp_1 + c^2p_1^2 && (11)
\end{aligned}
$$

Solving for cp_1 we get

$$
\begin{aligned}
cp_1 &= \frac{\left(mc^2 + \sqrt{m^2c^4 + p_e^2c^2}\right)^2 - c^2p_e^2}{2\left(mc^2 + \sqrt{m^2c^4 + p_e^2c^2}\right) - cp_e} && (12)\\
&= \frac{2m^2c^4 + 2mc^2\sqrt{m^2c^4 + p_e^2c^2}}{2\left(mc^2 + \sqrt{m^2c^4 + p_e^2c^2}\right) - cp_e} && (13)\\
&= \frac{mc^2}{1 - cp_e/\left[2\left(mc^2 + \sqrt{m^2c^4 + p_e^2c^2}\right)\right]} && (14)
\end{aligned}
$$

Thus photon 1 has an energy greater than the rest mass of the electron. Note that if the pair annihilation occurs with both the electron and positron at rest ($p_e = 0$), then $cp_1 = cp_2 = mc^2$ as we'd expect.