FOUR-ACCELERATION AND MINKOWSKI FORCE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.38.

We can define a *four-acceleration* as the derivative of four-velocity with respect to proper time:

$$\alpha^i \equiv \frac{d\eta^i}{d\tau} \tag{1}$$

where

$$\begin{aligned}
\eta^0 &= \frac{c}{\sqrt{1-u^2/c^2}} & (2)\\
\boldsymbol{\eta} &= \frac{\mathbf{u}}{\sqrt{1-u^2/c^2}} & (3)
\end{aligned}$$

Looking first at α^0, we have

$$\begin{aligned}
\alpha^0 &= \frac{d\eta^0}{d\tau} & (4)\\
&= c\frac{d}{d\tau}\left(\frac{1}{\sqrt{1-u^2/c^2}}\right) & (5)\\
&= c\frac{dt}{d\tau}\frac{d}{dt}\left(\frac{1}{\sqrt{1-u^2/c^2}}\right) & (6)\\
&= \frac{c}{\sqrt{1-u^2/c^2}}\frac{d}{dt}\left(\frac{1}{\sqrt{1-u^2/c^2}}\right) & (7)
\end{aligned}$$

The derivative in the last line we worked out earlier:

$$\frac{d}{dt}\left(\frac{1}{\sqrt{1-u^2/c^2}}\right) = \frac{\mathbf{u}\cdot\mathbf{a}}{c^2\left(1-u^2/c^2\right)^{3/2}} \tag{8}$$

so

$$\alpha^0 = \frac{\mathbf{u}\cdot\mathbf{a}}{c\left(1-u^2/c^2\right)^2} \tag{9}$$

The spatial components work out to

$$\begin{aligned} \boldsymbol{\alpha} &= \frac{1}{m}\frac{d\mathbf{p}}{d\tau} && (10)\\ &= \frac{1}{m\sqrt{1-u^2/c^2}}\frac{d\mathbf{p}}{dt} && (11)\\ &= \frac{1}{m\sqrt{1-u^2/c^2}}\mathbf{F} && (12) \end{aligned}$$

We alsoworked out the force earlier:

$$\mathbf{F} = \frac{m}{\sqrt{1-u^2/c^2}}\left[\mathbf{a} + \frac{(\mathbf{u}\cdot\mathbf{a})\,\mathbf{u}}{c^2-u^2}\right] \tag{13}$$

so

$$\boldsymbol{\alpha} = \frac{1}{1-u^2/c^2}\left[\mathbf{a} + \frac{(\mathbf{u}\cdot\mathbf{a})\,\mathbf{u}}{c^2-u^2}\right] \tag{14}$$

The invariant square of α^i is (using $\gamma \equiv 1/\sqrt{1-u^2/c^2}$)

$$\begin{aligned} \alpha_i\alpha^i &= -\gamma^8\frac{(\mathbf{u}\cdot\mathbf{a})^2}{c^2} + \gamma^4\left[a^2 + \frac{2\gamma^2(\mathbf{u}\cdot\mathbf{a})^2}{c^2} + \frac{\gamma^4u^2(\mathbf{u}\cdot\mathbf{a})^2}{c^4}\right] && (15)\\ &= \gamma^4a^2 + \frac{\gamma^8}{c^2}(\mathbf{u}\cdot\mathbf{a})^2\left[-1+\frac{2}{\gamma^2}+\frac{u^2}{c^2}\right] && (16)\\ &= \gamma^4a^2 + \frac{\gamma^8}{c^2}(\mathbf{u}\cdot\mathbf{a})^2\left[1-2\frac{u^2}{c^2}+\frac{u^2}{c^2}\right] && (17)\\ &= \gamma^4a^2 + \frac{\gamma^6}{c^2}(\mathbf{u}\cdot\mathbf{a})^2 && (18)\\ &= \frac{1}{\left(1-u^2/c^2\right)^2}\left[a^2+\frac{(\mathbf{u}\cdot\mathbf{a})^2}{c^2-u^2}\right] && (19) \end{aligned}$$

This reduces to a^2 to first order in u.
The scalar product of four-acceleration and four-velocity is, using 3:

$$\begin{aligned} \alpha^i \eta_i &= -\gamma^5 (\mathbf{u}\cdot\mathbf{a}) + \gamma^3 (\mathbf{u}\cdot\mathbf{a}) + \gamma^5 (\mathbf{u}\cdot\mathbf{a}) \frac{u^2}{c^2} && (20) \\ &= (\mathbf{u}\cdot\mathbf{a}) \gamma^3 \left[1 - \gamma^2 \left(1 - \frac{u^2}{c^2}\right)\right] && (21) \\ &= 0 && (22) \end{aligned}$$

Finally, we can use the four-acceleration to write the *Minkowski force*, which is the derivative of the four-momentum with respect to the proper time:

$$\begin{aligned} \mathbf{K} &\equiv \frac{d\mathbf{p}}{d\tau} && (23) \\ &= m\alpha && (24) \end{aligned}$$

The Minkowski force, used with four-acceleration, looks just like the non-relativistic form of Newton's second law. If we complete the Minkowski force by including its 0 component, we have

$$K^0 = m\alpha^0 \quad (25)$$

from which the invariant product with four-velocity follows:

$$K^i \eta_i = m\alpha^i \eta_i = 0 \quad (26)$$

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