

FOUR-ACCELERATION AND MINKOWSKI FORCE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.38.

We can define a *four-acceleration* as the derivative of four-velocity with respect to proper time:

$$(1) \quad \alpha^i \equiv \frac{d\eta^i}{d\tau}$$

where

$$(2) \quad \eta^0 = \frac{c}{\sqrt{1-u^2/c^2}}$$

$$(3) \quad \boldsymbol{\eta} = \frac{\mathbf{u}}{\sqrt{1-u^2/c^2}}$$

Looking first at α^0 , we have

$$(4) \quad \alpha^0 = \frac{d\eta^0}{d\tau}$$

$$(5) \quad = c \frac{d}{d\tau} \left(\frac{1}{\sqrt{1-u^2/c^2}} \right)$$

$$(6) \quad = c \frac{dt}{d\tau} \frac{d}{dt} \left(\frac{1}{\sqrt{1-u^2/c^2}} \right)$$

$$(7) \quad = \frac{c}{\sqrt{1-u^2/c^2}} \frac{d}{dt} \left(\frac{1}{\sqrt{1-u^2/c^2}} \right)$$

The derivative in the last line we worked out earlier:

$$(8) \quad \frac{d}{dt} \left(\frac{1}{\sqrt{1-u^2/c^2}} \right) = \frac{\mathbf{u} \cdot \mathbf{a}}{c^2 (1-u^2/c^2)^{3/2}}$$

so

$$(9) \quad \alpha^0 = \frac{\mathbf{u} \cdot \mathbf{a}}{c(1 - u^2/c^2)^2}$$

The spatial components work out to

$$(10) \quad \alpha = \frac{1}{m} \frac{d\mathbf{p}}{d\tau}$$

$$(11) \quad = \frac{1}{m\sqrt{1 - u^2/c^2}} \frac{d\mathbf{p}}{dt}$$

$$(12) \quad = \frac{1}{m\sqrt{1 - u^2/c^2}} \mathbf{F}$$

We also worked out the force earlier:

$$(13) \quad \mathbf{F} = \frac{m}{\sqrt{1 - u^2/c^2}} \left[\mathbf{a} + \frac{(\mathbf{u} \cdot \mathbf{a}) \mathbf{u}}{c^2 - u^2} \right]$$

so

$$(14) \quad \alpha = \frac{1}{1 - u^2/c^2} \left[\mathbf{a} + \frac{(\mathbf{u} \cdot \mathbf{a}) \mathbf{u}}{c^2 - u^2} \right]$$

The invariant square of α^i is (using $\gamma \equiv 1/\sqrt{1 - u^2/c^2}$)

$$(15) \quad \alpha_i \alpha^i = -\gamma^8 \frac{(\mathbf{u} \cdot \mathbf{a})^2}{c^2} + \gamma^4 \left[a^2 + \frac{2\gamma^2 (\mathbf{u} \cdot \mathbf{a})^2}{c^2} + \frac{\gamma^4 u^2 (\mathbf{u} \cdot \mathbf{a})^2}{c^4} \right]$$

$$(16) \quad = \gamma^4 a^2 + \frac{\gamma^8}{c^2} (\mathbf{u} \cdot \mathbf{a})^2 \left[-1 + \frac{2}{\gamma^2} + \frac{u^2}{c^2} \right]$$

$$(17) \quad = \gamma^4 a^2 + \frac{\gamma^8}{c^2} (\mathbf{u} \cdot \mathbf{a})^2 \left[1 - 2\frac{u^2}{c^2} + \frac{u^2}{c^2} \right]$$

$$(18) \quad = \gamma^4 a^2 + \frac{\gamma^6}{c^2} (\mathbf{u} \cdot \mathbf{a})^2$$

$$(19) \quad = \frac{1}{(1 - u^2/c^2)^2} \left[a^2 + \frac{(\mathbf{u} \cdot \mathbf{a})^2}{c^2 - u^2} \right]$$

This reduces to a^2 to first order in u .

The scalar product of four-acceleration and four-velocity is, using 3:

$$(20) \quad \alpha^i \eta_i = -\gamma^5 (\mathbf{u} \cdot \mathbf{a}) + \gamma^3 (\mathbf{u} \cdot \mathbf{a}) + \gamma^5 (\mathbf{u} \cdot \mathbf{a}) \frac{u^2}{c^2}$$

$$(21) \quad = (\mathbf{u} \cdot \mathbf{a}) \gamma^3 \left[1 - \gamma^2 \left(1 - \frac{u^2}{c^2} \right) \right]$$

$$(22) \quad = 0$$

Finally, we can use the four-acceleration to write the *Minkowski force*, which is the derivative of the four-momentum with respect to the proper time:

$$(23) \quad \mathbf{K} \equiv \frac{d\mathbf{p}}{d\tau}$$

$$(24) \quad = m\boldsymbol{\alpha}$$

The Minkowski force, used with four-acceleration, looks just like the non-relativistic form of Newton's second law. If we complete the Minkowski force by including its 0 component, we have

$$(25) \quad K^0 = m\alpha^0$$

from which the invariant product with four-velocity follows:

$$(26) \quad K^i \eta_i = m\alpha^i \eta_i = 0$$

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