RELATIVISTIC ELECTROMAGNETIC FORCE

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.40.

Ordinary force in relativity is given by

$$\mathbf{F}=\frac{m}{\sqrt{1-u^{2}/c^{2}}}\left[\mathbf{a}+\frac{(\mathbf{u}\cdot\mathbf{a})\,\mathbf{u}}{c^{2}-u^{2}}\right] \tag{1}$$

Suppose we have a particle of charge q travelling with velocity $\mathbf{u}$ in a region with electric and magnetic fields, so that the force is given by

$$\mathbf{F}=q\mathbf{E}+q\mathbf{u}\times\mathbf{B} \tag{2}$$

What is the acceleration of the particle? From 1 we have

$$\begin{aligned}
\mathbf{a} &= \frac{\sqrt{1-u^{2}/c^{2}}}{m}\mathbf{F}-\frac{(\mathbf{u}\cdot\mathbf{a})\,\mathbf{u}}{c^{2}-u^{2}} && (3)\\
&= \frac{q\sqrt{1-u^{2}/c^{2}}}{m}\left(\mathbf{E}+\mathbf{u}\times\mathbf{B}\right)-\frac{(\mathbf{u}\cdot\mathbf{a})\,\mathbf{u}}{c^{2}-u^{2}} && (4)\\
&= \frac{q}{\gamma m}\left(\mathbf{E}+\mathbf{u}\times\mathbf{B}\right)-\gamma^{2}\frac{(\mathbf{u}\cdot\mathbf{a})\,\mathbf{u}}{c^{2}} && (5)
\end{aligned}$$

The trick is to disentangle the $\mathbf{a}$ from the $\mathbf{u}\cdot\mathbf{a}$ term on the RHS. We can do this by taking the dot product of this equation with $\mathbf{u}$ to get

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{a} &= \frac{q\sqrt{1-u^{2}/c^{2}}}{m}\left(\mathbf{E}+\mathbf{u}\times\mathbf{B}\right)\cdot\mathbf{u}-\frac{(\mathbf{u}\cdot\mathbf{a})\,u^{2}}{c^{2}-u^{2}} && (6)\\
&= \frac{q\sqrt{1-u^{2}/c^{2}}}{m}\mathbf{E}\cdot\mathbf{u}-\frac{(\mathbf{u}\cdot\mathbf{a})\,u^{2}}{c^{2}-u^{2}} && (7)\\
&= \frac{q}{\gamma m}\mathbf{E}\cdot\mathbf{u}-\gamma^{2}\frac{(\mathbf{u}\cdot\mathbf{a})\,u^{2}}{c^{2}} && (8)
\end{aligned}$$

We can now solve for $\mathbf{u}\cdot\mathbf{a}$:

$$\mathbf{u}\cdot\mathbf{a} = \frac{q}{\gamma m}\mathbf{E}\cdot\mathbf{u}\left(1+\frac{u^2}{c^2}\gamma^2\right)^{-1} \tag{9}$$

$$= \frac{q}{\gamma^3 m}\mathbf{E}\cdot\mathbf{u} \tag{10}$$

Substituting back into 5 we get

$$\mathbf{a} = \frac{q}{\gamma m}\left(\mathbf{E}+\mathbf{u}\times\mathbf{B}-\frac{1}{c^2}\left(\mathbf{E}\cdot\mathbf{u}\right)\mathbf{u}\right) \tag{11}$$

$$= \frac{q}{m}\sqrt{1-u^2/c^2}\left(\mathbf{E}+\mathbf{u}\times\mathbf{B}-\frac{1}{c^2}\left(\mathbf{E}\cdot\mathbf{u}\right)\mathbf{u}\right) \tag{12}$$

Note that as $u\to c$, $\mathbf{a}\to 0$ no matter how strong the fields are, so again we're reminded that we can't accelerate any massive object up to the speed of light.