RELATIVISTIC ELECTROMAGNETIC FORCE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.40.

Ordinary force in relativity is given by

$$\mathbf{F}=\frac{m}{\sqrt{1-u^2/c^2}}\left[\mathbf{a}+\frac{(\mathbf{u}\cdot\mathbf{a})\,\mathbf{u}}{c^2-u^2}\right] \tag{1}$$

Suppose we have a particle of charge q travelling with velocity $\mathbf{u}$ in a region with electric and magnetic fields, so that the force is given by

$$\mathbf{F}=q\mathbf{E}+q\mathbf{u}\times\mathbf{B} \tag{2}$$

What is the acceleration of the particle? From 1 we have

$$\begin{aligned}
\mathbf{a} &= \frac{\sqrt{1-u^2/c^2}}{m}\mathbf{F}-\frac{(\mathbf{u}\cdot\mathbf{a})\,\mathbf{u}}{c^2-u^2} &(3)\\
&= \frac{q\sqrt{1-u^2/c^2}}{m}(\mathbf{E}+\mathbf{u}\times\mathbf{B})-\frac{(\mathbf{u}\cdot\mathbf{a})\,\mathbf{u}}{c^2-u^2} &(4)\\
&= \frac{q}{\gamma m}(\mathbf{E}+\mathbf{u}\times\mathbf{B})-\gamma^2\frac{(\mathbf{u}\cdot\mathbf{a})\,\mathbf{u}}{c^2} &(5)
\end{aligned}$$

The trick is to disentangle the $\mathbf{a}$ from the $\mathbf{u}\cdot\mathbf{a}$ term on the RHS. We can do this by taking the dot product of this equation with $\mathbf{u}$ to get

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{a} &= \frac{q\sqrt{1-u^2/c^2}}{m}(\mathbf{E}+\mathbf{u}\times\mathbf{B})\cdot\mathbf{u}-\frac{(\mathbf{u}\cdot\mathbf{a})\,u^2}{c^2-u^2} &(6)\\
&= \frac{q\sqrt{1-u^2/c^2}}{m}\mathbf{E}\cdot\mathbf{u}-\frac{(\mathbf{u}\cdot\mathbf{a})\,u^2}{c^2-u^2} &(7)\\
&= \frac{q}{\gamma m}\mathbf{E}\cdot\mathbf{u}-\gamma^2\frac{(\mathbf{u}\cdot\mathbf{a})\,u^2}{c^2} &(8)
\end{aligned}$$

We can now solve for $\mathbf{u}\cdot\mathbf{a}$:

$$\mathbf{u}\cdot\mathbf{a} = \frac{q}{\gamma m}\mathbf{E}\cdot\mathbf{u}\left(1+\frac{u^2}{c^2}\gamma^2\right)^{-1} \tag{9}$$

$$= \frac{q}{\gamma^3 m}\mathbf{E}\cdot\mathbf{u} \tag{10}$$

Substituting back into 5 we get

$$\mathbf{a} = \frac{q}{\gamma m}\left(\mathbf{E}+\mathbf{u}\times\mathbf{B}-\frac{1}{c^2}(\mathbf{E}\cdot\mathbf{u})\mathbf{u}\right) \tag{11}$$

$$= \frac{q}{m}\sqrt{1-u^2/c^2}\left(\mathbf{E}+\mathbf{u}\times\mathbf{B}-\frac{1}{c^2}(\mathbf{E}\cdot\mathbf{u})\mathbf{u}\right) \tag{12}$$

Note that as $u\to c$, $\mathbf{a}\to 0$ no matter how strong the fields are, so again we're reminded that we can't accelerate any massive object up to the speed of light.