

## RELATIVISTIC ELECTROMAGNETIC FORCE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.40.

Ordinary force in relativity is given by

$$(0.1) \quad \mathbf{F} = \frac{m}{\sqrt{1-u^2/c^2}} \left[ \mathbf{a} + \frac{(\mathbf{u} \cdot \mathbf{a}) \mathbf{u}}{c^2 - u^2} \right]$$

Suppose we have a particle of charge  $q$  travelling with velocity  $\mathbf{u}$  in a region with electric and magnetic fields, so that the force is given by

$$(0.2) \quad \mathbf{F} = q\mathbf{E} + q\mathbf{u} \times \mathbf{B}$$

What is the acceleration of the particle? From 0.1 we have

$$(0.3) \quad \mathbf{a} = \frac{\sqrt{1-u^2/c^2}}{m} \mathbf{F} - \frac{(\mathbf{u} \cdot \mathbf{a}) \mathbf{u}}{c^2 - u^2}$$

$$(0.4) \quad = \frac{q\sqrt{1-u^2/c^2}}{m} (\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \frac{(\mathbf{u} \cdot \mathbf{a}) \mathbf{u}}{c^2 - u^2}$$

$$(0.5) \quad = \frac{q}{\gamma m} (\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \gamma^2 \frac{(\mathbf{u} \cdot \mathbf{a}) \mathbf{u}}{c^2}$$

The trick is to disentangle the  $\mathbf{a}$  from the  $\mathbf{u} \cdot \mathbf{a}$  term on the RHS. We can do this by taking the dot product of this equation with  $\mathbf{u}$  to get

$$(0.6) \quad \mathbf{u} \cdot \mathbf{a} = \frac{q\sqrt{1-u^2/c^2}}{m} (\mathbf{E} + \mathbf{u} \times \mathbf{B}) \cdot \mathbf{u} - \frac{(\mathbf{u} \cdot \mathbf{a}) u^2}{c^2 - u^2}$$

$$(0.7) \quad = \frac{q\sqrt{1-u^2/c^2}}{m} \mathbf{E} \cdot \mathbf{u} - \frac{(\mathbf{u} \cdot \mathbf{a}) u^2}{c^2 - u^2}$$

$$(0.8) \quad = \frac{q}{\gamma m} \mathbf{E} \cdot \mathbf{u} - \gamma^2 \frac{(\mathbf{u} \cdot \mathbf{a}) u^2}{c^2}$$

We can now solve for  $\mathbf{u} \cdot \mathbf{a}$ :

$$(0.9) \quad \mathbf{u} \cdot \mathbf{a} = \frac{q}{\gamma m} \mathbf{E} \cdot \mathbf{u} \left( 1 + \frac{u^2}{c^2} \gamma^2 \right)^{-1}$$

$$(0.10) \quad = \frac{q}{\gamma^3 m} \mathbf{E} \cdot \mathbf{u}$$

Substituting back into 0.5 we get

$$(0.11) \quad \mathbf{a} = \frac{q}{\gamma m} \left( \mathbf{E} + \mathbf{u} \times \mathbf{B} - \frac{1}{c^2} (\mathbf{E} \cdot \mathbf{u}) \mathbf{u} \right)$$

$$(0.12) \quad = \frac{q}{m} \sqrt{1 - u^2/c^2} \left( \mathbf{E} + \mathbf{u} \times \mathbf{B} - \frac{1}{c^2} (\mathbf{E} \cdot \mathbf{u}) \mathbf{u} \right)$$

Note that as  $u \rightarrow c$ ,  $\mathbf{a} \rightarrow 0$  no matter how strong the fields are, so again we're reminded that we can't accelerate any massive object up to the speed of light.