RELATIVISTIC TRANSFORMATION OF FORCE, ELECTRIC AND MAGNETIC FIELDS

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problems 12.41, 12.44.

Griffiths gives fairly complete derivations of the relativistic transformation laws for force and electromagnetic fields as we move from one inertial frame to another in his sections 12.2.4 and 12.3.2, so I won't grind through the whole thing again here. I'll just recap the ideas behind these derivations to give a flavour of what's being done.

'Ordinary' force is defined as the derivative of the four-momentum (or at least its spatial part) with respect to 'ordinary' (not proper) time. Since four-momentum is a four-vector, it transforms using Lorentz transformations, as does 'ordinary' time. The problem with force is that its transformation is then the ratio of two Lorentz-transformed objects, so the transformation equations get a bit messy. The derivation of the transformations is done in much the same way as the derivation of the velocity addition formulas. For a frame $\bar{\mathcal{S}}$ moving at velocity v in the x direction relative to a stationary frame $\mathcal{S}$ we get

$$\begin{aligned}
\bar{F}_y &= \frac{d\bar{p}_y}{d\bar{t}} && (1)\\
&= \frac{dp_y}{\gamma\left(dt-\frac{\beta}{c}dx\right)} && (2)\\
&= \frac{dp_y/dt}{\gamma(1-\beta u_x/c)} && (3)\\
&= \frac{F_y}{\gamma(1-\beta u_x/c)} && (4)
\end{aligned}$$

where $\mathbf{u}$ is the object's velocity in $\mathcal{S}$. Remember that the velocity v used in the calculation of γ and β is the relative velocity of $\mathcal{S}$ and $\bar{\mathcal{S}}$ and $\mathbf{u}$ is the velocity of the object relative to $\mathcal{S}$.

The calculation of the other two components is done in a similar way, with the result

$$\begin{aligned}
\bar{F}_x &= \frac{F_x - \beta\left(\mathbf{u}\cdot\mathbf{F}\right)/c}{1-\beta u_x/c} && (5)\\
\bar{F}_y &= \frac{F_y}{\gamma(1-\beta u_x/c)} && (6)\\
\bar{F}_z &= \frac{F_z}{\gamma(1-\beta u_x/c)} && (7)
\end{aligned}$$

Example 1. We have a charge q_A at rest at the origin of $\mathcal{S}$ and another charge q_B moving at speed v in the $+x$ direction along the line $y=d$. When q_B crosses the y axis, it feels only an electric field (since q_A is at rest it generates no magnetic field in $\mathcal{S}$), so the force on it is

$$\mathbf{F} = \frac{q_A q_B}{4\pi\varepsilon_0 d^2}\hat{\mathbf{y}} \tag{8}$$

If we now switch to q_B's frame $\bar{\mathcal{S}}$ which is moving relative to $\mathcal{S}$ at speed v along the x axis, we can find the force experienced by q_B in this frame using 6. In this case, $\mathbf{u}=\mathbf{v}$ (the object's velocity in $\mathcal{S}$ is the same as the relative velocity of $\mathcal{S}$ and $\bar{\mathcal{S}}$) so

$$\begin{aligned}
\bar{F}_y &= \frac{F_y}{\gamma(1-\beta v/c)} && (9)\\
&= \frac{F_y}{\gamma(1-v^2/c^2)} && (10)\\
&= \gamma F_y && (11)\\
&= \frac{\gamma q_A q_B}{4\pi\varepsilon_0 d^2}\hat{\mathbf{y}} && (12)
\end{aligned}$$

That is, the force experienced by q_B is greater in the frame at which it is at rest.

In fact we can see from the transformation equations above that all components of force perpendicular to the motion are at a maximum in an object's rest frame. In that frame $\mathbf{u}=0$ so the equations become

$$\begin{aligned}
\bar{F}_x &= F_x && (13)\\
\bar{F}_y &= \frac{F_y}{\gamma} && (14)\\
\bar{F}_z &= \frac{F_z}{\gamma} && (15)
\end{aligned}$$

To transform the electric and magnetic fields we can use a similar approach to that for the electric field, in which we considered a parallel plate capacitor with charge densities of $\pm\sigma_0$ (in the capacitor's rest frame $\mathcal{S}_0$) on the two plates. In $\mathcal{S}_0$ however, since the charge is at rest, there is no magnetic field so we can't use that system as it stands to derive the general transformation rules for electromagnetic fields. What we can do is introduce two other frames $\mathcal{S}$ (moving at speed v_0 relative to $\mathcal{S}_0$) and $\bar{\mathcal{S}}$ (moving at speed v relative to $\mathcal{S}$). From the velocity addition formula, the velocity of $\bar{\mathcal{S}}$ relative to $\mathcal{S}_0$ is then

$$\bar{v} = \frac{v+v_0}{1+vv_0/c^2} \tag{16}$$

Since we now have the velocities of both $\mathcal{S}$ and $\bar{\mathcal{S}}$ relative to $\mathcal{S}_0$ (where the charge is at rest, remember), we can eliminate $\mathcal{S}_0$ and express the fields in $\bar{\mathcal{S}}$ in terms of those in $\mathcal{S}$. Griffiths goes through the details, with the results

$$\begin{aligned}
\bar{E}_x &= E_x && (17)\\
\bar{E}_y &= \gamma(E_y - vB_z) && (18)\\
\bar{E}_z &= \gamma(E_z + vB_y) && (19)\\
\bar{B}_x &= B_x && (20)\\
\bar{B}_y &= \gamma\left(B_y + \frac{v}{c^2}E_z\right) && (21)\\
\bar{B}_z &= \gamma\left(B_z - \frac{v}{c^2}E_y\right) && (22)
\end{aligned}$$

Notice how the electric and magnetic field components perpendicular to the motion get tangled up with each other when we transform frames. This shows how it's possible for a test charge to experience only an electric field in one frame, but a combination of electric and magnetic fields in another frame (and vice versa).

Example 2. Now that we have the general transformation rules, we can return to the special case of frames $\mathcal{S}$ in which the capacitor plates are at rest and $\bar{\mathcal{S}}$ where they are moving in the x direction at speed v. When the plates are at rest, there is only an electric field (in the y direction, assuming that the plates are parallel to the xz plane). From these equations, we see that in any other frame moving in the x direction, the electric field will have only a y component and the magnetic field will have only a z component. [I'm not certain this is the approach Griffiths wants in his problem 12.41 in which he asks why $\bar{E}_z = 0$ in the moving frame, but it does appear to answer the question.]

Example 3. We can also revisit Example 1 and calculate the force felt by q_B in its own rest frame by applying the field transformations. When q_B

crosses the $\bar{y}$ axis (which we'll assume coincides with the y axis at the time when q_B crosses it) we had, from above,

$$E_y = \frac{q_A}{4\pi\varepsilon_0 d^2} \tag{23}$$

with $E_x = E_z = 0$, and $\mathbf{B} = 0$. In q_B's frame, we have

$$\bar{E}_y = \gamma E_y = \frac{\gamma q_A}{4\pi\varepsilon_0 d^2} \tag{24}$$

$$\bar{B}_z = -\gamma \frac{v}{c^2} E_y = -\frac{\gamma q_A v}{4\pi\varepsilon_0 d^2 c^2} \tag{25}$$

with all other components equal to zero. The force felt by q_B is again entirely electric (since it's not moving in its own rest frame, it experiences no magnetic force), and we get

$$\bar{F}_y = q_B \bar{E}_y = \frac{\gamma q_A q_B}{4\pi\varepsilon_0 d^2} \tag{26}$$

which is the same as 12.

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