TRANSFORMING THE ELECTRIC FIELD IN RELATIVITY

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.42.

Applying relativity to electromagnetism results in some rather curious behaviour. For example, suppose we have a wire at rest in the lab frame. The wire contains a linear charge density λ of positive charge and a density $-\lambda$ of negative charge. Since the densities are equal and opposite, the wire is electrically neutral. Now suppose we start the positive charge moving with speed v in the $+x$ direction, and the negative charge moving with equal and opposite speed in the $-x$ direction, resulting in a total net current of $I = 2\lambda v$ in the $+x$ direction. This produces a magnetic field (using Ampère's law) at a distance r from the wire of

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi} = \frac{\mu_0 \lambda v}{\pi r}\hat{\phi} \tag{1}$$

If we now place a charge $+q$ a distance r from the wire moving a velocity u in the $+x$ direction (parallel to the wire), it will experience no electric force (since the wire is neutral) but it will experience a magnetic force of

$$\mathbf{F} = q\mathbf{u}\times\mathbf{B} = -qu\frac{\mu_0 \lambda v}{\pi r}\hat{\mathbf{r}} \tag{2}$$

That is, it experiences a force towards the wire.

Now suppose we switch to a frame moving with velocity u in the $+x$ direction, in which q is at rest. Since the charge is at rest, it experiences no magnetic force and, since the wire is neutral, we might think it experiences no electric force either. This conclusion violates the first principle of relativity, namely that the laws of physics should appear the same in all inertial frames. In the first frame (with moving q), the charge experiences a force towards the wire, while in the second frame (charge at rest) it experiences no force at all. What went wrong?

To solve this problem, we can apply relativity to the transformation between frames. In the first frame, with the wire at rest, the conclusions about the electrical neutrality and currents are valid since we're not moving. Although the two lines of charge are both moving and are therefore Lorentz-contracted, their speeds are equal so they are both contracted by the same factor. However, when we transform to the second frame, the velocities of the two anti-parallel currents transform in different ways due to the velocity addition formula. The velocities of the two lines of charge in this second frame are (the subscript refers to the sign of the charge):

$$\bar{v}_{\pm}=\frac{v\mp u}{1\mp uv/c^{2}} \tag{3}$$

Because the positive charge is now seen as moving more slowly, it will not be Lorentz-contracted as much as the negative charge, so that in this frame, $\lambda_{+}<\lambda_{-}$. In other words, as viewed in the rest frame of the charge q, the wire has a net negative charge so q feels an *electric* (not magnetic) force of attraction to the wire. In his section 12.3.1, Griffiths goes through the algebra to show that this electric force is equal to the magnetic force felt in the first frame (after appropriately transforming the force between the two frames).

We can use a similar argument to work out how the electric field transforms between frames. This time, we start out with a large parallel plate capacitor with its plates parallel to the xz plane and separated by a distance d. If the upper plate has surface charge density $+\sigma_{0}$ and the lower plate has density $-\sigma_{0}$, then the electric field between the plates is, in the frame where the capacitor is at rest:

$$\mathbf{E}_{0}=\frac{\sigma_{0}}{\epsilon_{0}}\hat{\mathbf{y}} \tag{4}$$

Now suppose we move to a frame moving at velocity v in the $+x$ direction. The capacitor plates are now contracted by a factor $1/\gamma$ in the x direction, but their size is unchanged in the z direction since this direction is perpendicular to the motion. Therefore, the area of the plates is reduced by a factor $1/\gamma$ so the charge density increases by a factor γ resulting in an electric field

$$\mathbf{E}=\gamma\frac{\sigma_{0}}{\epsilon_{0}}\hat{\mathbf{y}}=\gamma\mathbf{E}_{0} \tag{5}$$

Clearly the same logic applies if we oriented the plates parallel to the xy plane (the electric field would then be in the z direction and would increase by the same factor γ), so in general we can say that

$$\mathbf{E}^{\perp}=\gamma\mathbf{E}_{0}^{\perp} \tag{6}$$

That is, the electric field perpendicular to the motion increases by a factor γ.

To get the behaviour of the field parallel to the motion, we can orient the plates parallel to the yz plane. This time the plates are perpendicular to the motion so their size does not change; however the distance d between the plates is contracted. Since the electric field doesn't depend on d (we're assuming that the plates are very large compared to their separation so we can ignore edge effects) the parallel components of **E** are unchanged:

$$\mathbf{E}^{\parallel}=\mathbf{E}_0^{\parallel} \tag{7}$$

Notice that we've implicitly assumed that the charge is invariant under a Lorentz transformation. The theoretical reasons for this appear to be rather deep, although it seems that if we require Maxwell's equations to be Lorentz-invariant, then charge must also be Lorentz-invariant. In any case, it's a bit more than I want to get into here.

Example. Suppose we now tilt the capacitor plates so that they make an angle of $\frac{\pi}{4}$ with the x axis. A vector lying in the $x-y$ plane and in the plane of the plates is $\hat{\mathbf{x}}+\hat{\mathbf{y}}$, so that the normal vector to the plates is

$$\mathbf{n}_0=-\hat{\mathbf{x}}+\hat{\mathbf{y}} \tag{8}$$

In the plates' rest frame, the field is

$$\mathbf{E}_0=\frac{E_0}{\sqrt{2}}\left(-\hat{\mathbf{x}}+\hat{\mathbf{y}}\right) \tag{9}$$

If we now shift to a frame moving at velocity v, only the y component of $\mathbf{E}_0$ is affected, so the new field is

$$\mathbf{E}=\frac{E_0}{\sqrt{2}}\left(-\hat{\mathbf{x}}+\gamma\hat{\mathbf{y}}\right) \tag{10}$$

However, the motion also contracts the x dimension of the plates by a factor γ, so a vector lying in the $x-y$ plane and in the plane of the plates is now $\frac{1}{\gamma}\hat{\mathbf{x}}+\hat{\mathbf{y}}$

making the new normal vector to the plates

$$\mathbf{n}=-\gamma\hat{\mathbf{x}}+\hat{\mathbf{y}} \tag{11}$$

Therefore, in the moving frame, the angle between **E** and **n** is given by

$$\cos\theta=\frac{\mathbf{n}\cdot\mathbf{E}}{|\mathbf{n}|\,|\mathbf{E}|}=\frac{2\gamma}{1+\gamma^2} \tag{12}$$

For $\gamma>1$, $\cos\theta\neq 1$ so for the moving frame, $\theta\neq 0$ and the field is not perpendicular to the plates.

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