GAUSS'S LAW FOR A RELATIVISTIC POINT CHARGE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.43.

In Griffiths's example 12.13, he rederives the formula for the electric field due to a moving charge, this time using relativity instead of retarded potentials. The result, which we've examined before, is

$$\begin{aligned}
\mathbf{E}(\mathbf{r},t) &= \frac{q}{4\pi\varepsilon_0}\frac{1-v^2/c^2}{\left(1-v^2\sin^2\theta/c^2\right)^{3/2}}\frac{\hat{\mathbf{R}}}{R^2} && (1)\\
&= \frac{q}{4\pi\varepsilon_0\gamma^2}\frac{1}{\left(1-v^2\sin^2\theta/c^2\right)^{3/2}}\frac{\hat{\mathbf{R}}}{R^2} && (2)\\
&= \frac{q}{4\pi\varepsilon_0\gamma^2}\frac{1}{\left(1-\beta^2\sin^2\theta\right)^{3/2}}\frac{\hat{\mathbf{R}}}{R^2} && (3)
\end{aligned}$$

where

$$\mathbf{R}\equiv\mathbf{r}-\mathbf{v}t \tag{4}$$

is the vector from the particle's present (not retarded) position to the observer (assuming the particle passes through the origin at $t=0$) and θ is the angle between $\mathbf{R}$ and $\mathbf{v}$. We can verify that Gauss's law holds for this moving charge by integrating $\mathbf{E}\cdot d\mathbf{a}$ over a sphere of radius R.

$$\int\mathbf{E}\cdot d\mathbf{a}=\frac{q}{2\varepsilon_0\gamma^2}\int_0^{\pi}\frac{\sin\theta d\theta}{\left(1-v^2\sin^2\theta/c^2\right)^{3/2}} \tag{5}$$

The integral is nasty because we're missing a $\cos\theta$ in the numerator that would make the integral easy. Maple handles it easily enough, but for those interested in how to derive it, I worked backward from Maple's answer to figure out how to do it. First, we can split the integrand into the sum of two terms (I'll use $\beta\equiv v/c$ to simplify the notation):

$$\frac{\sin\theta}{\left(1-\beta^2\sin^2\theta\right)^{3/2}} = \frac{\sin\theta}{\left(1-\beta^2+\beta^2\cos^2\theta\right)^{3/2}} \tag{6}$$

$$= \frac{1}{(1-\beta^2)}\frac{\sin\theta\left(1-\beta^2+\beta^2\cos^2\theta-\beta^2\cos^2\theta\right)}{\left(1-\beta^2+\beta^2\cos^2\theta\right)^{3/2}} \tag{7}$$

$$= \frac{1}{(1-\beta^2)}\frac{\sin\theta}{\sqrt{1-\beta^2+\beta^2\cos^2\theta}} - \frac{\beta^2}{(1-\beta^2)}\frac{\cos^2\theta\sin\theta}{\left(1-\beta^2+\beta^2\cos^2\theta\right)^{3/2}} \tag{8}$$

We can now integrate the second term by parts:

$$-\frac{\beta^2}{(1-\beta^2)}\int\frac{\cos^2\theta\sin\theta}{\left(1-\beta^2+\beta^2\cos^2\theta\right)^{3/2}}d\theta = -\frac{\beta^2}{(1-\beta^2)}\int(\cos\theta)\left[\frac{\cos\theta\sin\theta}{\left(1-\beta^2+\beta^2\cos^2\theta\right)^{3/2}}\right]d\theta \tag{9}$$

$$= -\frac{\cos\theta}{(1-\beta^2)\sqrt{1-\beta^2+\beta^2\cos^2\theta}} - \frac{1}{(1-\beta^2)}\int\frac{\mathrm{si}}{\sqrt{1-\beta^2-}} \tag{10}$$

$$= -\frac{1}{(1-\beta^2)}\frac{\cos\theta}{\sqrt{1-\beta^2\sin^2\theta}} - \frac{1}{(1-\beta^2)}\int\frac{\sin\theta}{\sqrt{1-\beta^2+\beta^2\mathrm{c}}} \tag{11}$$

The second term in the last line now cancels the integral of the first term in 8, so we're left with

$$\int_0^\pi\frac{\sin\theta d\theta}{\left(1-v^2\sin^2\theta/c^2\right)^{3/2}} = -\frac{1}{(1-\beta^2)}\left.\frac{\cos\theta}{\sqrt{1-\beta^2\sin^2\theta}}\right|_0^\pi \tag{12}$$

$$= \frac{2}{1-\beta^2} = 2\gamma^2 \tag{13}$$

[There might be an easier way to do this, but I couldn't see any obvious substitutions that worked.]

Plugging this back into 5 we get

$$\int\mathbf{E}\cdot d\mathbf{a} = \frac{q}{\varepsilon_0} \tag{14}$$

so Gauss's law is satisfied.

We can also calculate the Poynting vector by using the magnetic field of a point charge in uniform motion, which Griffiths works out in his example 12.14:

$$\mathbf{B} = -\frac{1}{c^2}\mathbf{v}\times\mathbf{E} \tag{15}$$

$$= \frac{\mu_0}{4\pi}\frac{qv\left(1-v^2/c^2\right)\sin\theta}{\left[1-(v^2/c^2)\sin^2\theta\right]^{3/2}}\frac{\hat{\phi}}{R^2} \tag{16}$$

$$= \frac{\mu_0 qc\beta\sin\theta}{4\pi\gamma^2\left(1-\beta^2\sin^2\theta\right)^{3/2}}\frac{\hat{\phi}}{R^2} \tag{17}$$

where the ϕ direction is that found by using the right-hand rule with the thumb pointing in the direction of $\mathbf{v}$, the particle's motion. The Poynting vector is

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \tag{18}$$

If we restrict ourselves to a charge moving in the $+z$ direction, so that $\mathbf{v} = v\hat{\mathbf{z}}$ and at the time when the charge passes through the origin, then $\mathbf{R}$ in 3 becomes the radial coordinate in spherical coordinates and ϕ is the azimuthal coordinate. In that case, $\hat{\mathbf{R}}\times\hat{\phi} = -\hat{\theta}$ so

$$\mathbf{S} = -\frac{q^2}{16\pi^2\varepsilon_0\gamma^4}\frac{c\beta\sin\theta}{\left(1-\beta^2\sin^2\theta\right)^3}\frac{\hat{\theta}}{R^4} \tag{19}$$

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