

GAUSS'S LAW FOR A RELATIVISTIC POINT CHARGE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.43.

In Griffiths's example 12.13, he rederives the formula for the electric field due to a moving charge, this time using relativity instead of retarded potentials. The result, which we've examined before, is

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{(1 - v^2 \sin^2 \theta/c^2)^{3/2}} \frac{\hat{\mathbf{R}}}{R^2} \quad (1)$$

$$= \frac{q}{4\pi\epsilon_0\gamma^2} \frac{1}{(1 - v^2 \sin^2 \theta/c^2)^{3/2}} \frac{\hat{\mathbf{R}}}{R^2} \quad (2)$$

$$= \frac{q}{4\pi\epsilon_0\gamma^2} \frac{1}{(1 - \beta^2 \sin^2 \theta)^{3/2}} \frac{\hat{\mathbf{R}}}{R^2} \quad (3)$$

where

$$\mathbf{R} \equiv \mathbf{r} - \mathbf{v}t \quad (4)$$

is the vector from the particle's present (not retarded) position to the observer (assuming the particle passes through the origin at $t = 0$) and θ is the angle between \mathbf{R} and \mathbf{v} . We can verify that Gauss's law holds for this moving charge by integrating $\mathbf{E} \cdot d\mathbf{a}$ over a sphere of radius R .

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{q}{2\epsilon_0\gamma^2} \int_0^\pi \frac{\sin \theta d\theta}{(1 - v^2 \sin^2 \theta/c^2)^{3/2}} \quad (5)$$

The integral is nasty because we're missing a $\cos \theta$ in the numerator that would make the integral easy. Maple handles it easily enough, but for those interested in how to derive it, I worked backward from Maple's answer to figure out how to do it. First, we can split the integrand into the sum of two terms (I'll use $\beta \equiv v/c$ to simplify the notation):

$$\frac{\sin \theta}{(1 - \beta^2 \sin^2 \theta)^{3/2}} = \frac{\sin \theta}{(1 - \beta^2 + \beta^2 \cos^2 \theta)^{3/2}} \quad (6)$$

$$= \frac{1}{(1 - \beta^2)} \frac{\sin \theta (1 - \beta^2 + \beta^2 \cos^2 \theta - \beta^2 \cos^2 \theta)}{(1 - \beta^2 + \beta^2 \cos^2 \theta)^{3/2}} \quad (7)$$

$$= \frac{1}{(1 - \beta^2)} \frac{\sin \theta}{\sqrt{1 - \beta^2 + \beta^2 \cos^2 \theta}} - \frac{\beta^2}{(1 - \beta^2)} \frac{\cos^2 \theta \sin \theta}{(1 - \beta^2 + \beta^2 \cos^2 \theta)^{3/2}} \quad (8)$$

We can now integrate the second term by parts:

$$-\frac{\beta^2}{(1 - \beta^2)} \int \frac{\cos^2 \theta \sin \theta}{(1 - \beta^2 + \beta^2 \cos^2 \theta)^{3/2}} d\theta = -\frac{\beta^2}{(1 - \beta^2)} \int (\cos \theta) \left[\frac{\cos \theta \sin \theta}{(1 - \beta^2 + \beta^2 \cos^2 \theta)^{3/2}} \right] d\theta \quad (9)$$

$$= -\frac{\cos \theta}{(1 - \beta^2) \sqrt{1 - \beta^2 + \beta^2 \cos^2 \theta}} - \frac{1}{(1 - \beta^2)} \int \frac{\sin \theta}{\sqrt{1 - \beta^2 + \beta^2 \cos^2 \theta}} d\theta \quad (10)$$

$$= -\frac{1}{(1 - \beta^2)} \frac{\cos \theta}{\sqrt{1 - \beta^2 \sin^2 \theta}} - \frac{1}{(1 - \beta^2)} \int \frac{\sin \theta}{\sqrt{1 - \beta^2 + \beta^2 \cos^2 \theta}} d\theta \quad (11)$$

The second term in the last line now cancels the integral of the first term in 8, so we're left with

$$\int_0^\pi \frac{\sin \theta d\theta}{(1 - v^2 \sin^2 \theta / c^2)^{3/2}} = -\frac{1}{(1 - \beta^2)} \frac{\cos \theta}{\sqrt{1 - \beta^2 \sin^2 \theta}} \Big|_0^\pi \quad (12)$$

$$= \frac{2}{1 - \beta^2} = 2\gamma^2 \quad (13)$$

[There might be an easier way to do this, but I couldn't see any obvious substitutions that worked.]

Plugging this back into 5 we get

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{q}{\epsilon_0} \quad (14)$$

so Gauss's law is satisfied.

We can also calculate the Poynting vector by using the magnetic field of a point charge in uniform motion, which Griffiths works out in his example 12.14:

$$\mathbf{B} = -\frac{1}{c^2} \mathbf{v} \times \mathbf{E} \quad (15)$$

$$= \frac{\mu_0}{4\pi} \frac{qv(1-v^2/c^2) \sin \theta}{[1-(v^2/c^2) \sin^2 \theta]^{3/2}} \frac{\hat{\phi}}{R^2} \quad (16)$$

$$= \frac{\mu_0 qc\beta \sin \theta}{4\pi\gamma^2 (1-\beta^2 \sin^2 \theta)^{3/2}} \frac{\hat{\phi}}{R^2} \quad (17)$$

where the $\hat{\phi}$ direction is that found by using the right-hand rule with the thumb pointing in the direction of \mathbf{v} , the particle's motion. The Poynting vector is

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \quad (18)$$

If we restrict ourselves to a charge moving in the $+z$ direction, so that $\mathbf{v} = v\hat{\mathbf{z}}$ and at the time when the charge passes through the origin, then \mathbf{R} in 3 becomes the radial coordinate in spherical coordinates and ϕ is the azimuthal coordinate. In that case, $\hat{\mathbf{R}} \times \hat{\phi} = -\hat{\theta}$ so

$$\mathbf{S} = -\frac{q^2}{16\pi^2 \epsilon_0 \gamma^4} \frac{c\beta \sin \theta}{(1-\beta^2 \sin^2 \theta)^3} \frac{\hat{\theta}}{R^4} \quad (19)$$

PINGBACKS

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