

## RELATIVISTIC TRANSFORMATION OF ELECTRIC AND MAGNETIC FIELDS: AN EXAMPLE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.45.

As an example of the transformation equations for electromagnetic fields, consider the following situation. In the lab frame  $A$ , we have a charge  $-q$  moving at speed  $v$  in the  $+x$  direction, and a charge  $+q$  moving at the same speed  $v$  but in the  $-x$  direction, with  $-q$  following the path  $y = 0$  and  $+q$  following  $y = +d$ . Their positions are such that their closest approach occurs when they cross the  $y$  axis.

First, we can work out the fields and the force on  $+q$  at this point in the lab frame. The fields produced by a moving point charge are

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0\gamma^2} \frac{1}{(1 - \beta^2 \sin^2 \theta)^{3/2}} \frac{\hat{\mathbf{R}}}{R^2} \quad (1)$$

$$\mathbf{B} = \frac{1}{4\pi\epsilon_0 c^2} \frac{qv(1 - v^2/c^2) \sin \theta}{[1 - (v^2/c^2) \sin^2 \theta]^{3/2}} \frac{\hat{\phi}}{R^2} \quad (2)$$

where  $\mathbf{R}$  is the vector from the moving charge to the observer and the direction of  $\hat{\phi}$  is found from using the right-hand rule on the particle's velocity  $\mathbf{v}$ , as usual. The angle  $\theta$  is the angle between  $\mathbf{R}$  and  $\mathbf{v}$ . In our case, at the point where the charges are at their closest approach  $\theta = \pi/2$  and  $R = d$  so we get

$$\mathbf{E} = -\frac{q\gamma}{4\pi\epsilon_0 d^2} \hat{\mathbf{y}} \quad (3)$$

$$\mathbf{B} = -\frac{q\gamma v}{4\pi\epsilon_0 c^2 d^2} \hat{\mathbf{z}} \quad (4)$$

The force on  $+q$  with velocity  $\mathbf{v} = -v\hat{\mathbf{x}}$  can be found from the Lorentz force law:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (5)$$

$$= -\frac{q^2\gamma}{4\pi\epsilon_0 d^2} \left( \hat{\mathbf{y}} + \frac{v^2}{c^2} (-\hat{\mathbf{x}} \times \hat{\mathbf{z}}) \right) \quad (6)$$

$$= -\frac{q^2\gamma}{4\pi\epsilon_0 d^2} \left( 1 + \frac{v^2}{c^2} \right) \hat{\mathbf{y}} \quad (7)$$

Now suppose we switch to frame  $B$  in which  $+q$  is at rest. This frame is moving with velocity  $-v\hat{\mathbf{x}}$  with respect to  $A$ , so the transformation equations are

$$\bar{E}_x = E_x \quad (8)$$

$$\bar{E}_y = \gamma(E_y + vB_z) \quad (9)$$

$$\bar{E}_z = \gamma(E_z - vB_y) \quad (10)$$

$$\bar{B}_x = B_x \quad (11)$$

$$\bar{B}_y = \gamma\left(B_y - \frac{v}{c^2}E_z\right) \quad (12)$$

$$\bar{B}_z = \gamma\left(B_z + \frac{v}{c^2}E_y\right) \quad (13)$$

[Note that this is a special case where the speed  $v$  of the frame  $B$  happens to be the same as the speed of the charge in the original lab frame  $A$ , so we can use the same symbol for both. In the more general case, the  $v$  in the above 6 equations would be different from the  $v$  in equations 1 and 2.]

Only  $E_y$  and  $B_z$  are non-zero, so we get

$$\bar{\mathbf{E}} = -\frac{q\gamma^2}{4\pi\epsilon_0 d^2} \left( 1 + \frac{v^2}{c^2} \right) \hat{\mathbf{y}} \quad (14)$$

$$\bar{\mathbf{B}} = -\frac{2q\gamma^2 v}{4\pi\epsilon_0 d^2 c^2} \hat{\mathbf{z}} \quad (15)$$

[The  $y$  and  $\bar{y}$ , and  $z$  and  $\bar{z}$  axes are parallel so we can use the unit vectors from frame  $A$  or  $B$ .] The force seen in frame  $B$  (where the velocity of  $+q$  is  $\mathbf{v} = 0$  so there is no magnetic force) is thus

$$\bar{\mathbf{F}} = q\bar{\mathbf{E}} \quad (16)$$

$$= -\frac{q^2\gamma^2}{4\pi\epsilon_0 d^2} \left( 1 + \frac{v^2}{c^2} \right) \hat{\mathbf{y}} \quad (17)$$

Note that the force in frame  $B$  is larger by a factor of  $\gamma$  than the force in frame  $A$ . As we saw earlier, an object experiences its maximum force in the frame in which it's at rest.

Finally, let's look at things in frame  $C$  where  $-q$  is at rest. This can be found from the results from frame  $A$  by transforming to a frame moving at  $+v$  relative to  $A$ , so the transformation equations are equations 9 and 13 with  $v$  replaced by  $-v$ , giving

$$\bar{\mathbf{E}} = \gamma(E_y - vB_z)\hat{\mathbf{y}} \quad (18)$$

$$= -\frac{q\gamma^2}{4\pi\epsilon_0 d^2} \left(1 - \frac{v^2}{c^2}\right) \hat{\mathbf{y}} \quad (19)$$

$$= -\frac{q}{4\pi\epsilon_0 d^2} \hat{\mathbf{y}} \quad (20)$$

$$\bar{\mathbf{B}} = \gamma\left(B_z - \frac{v}{c^2}E_y\right)\hat{\mathbf{z}} \quad (21)$$

$$= 0 \quad (22)$$

Since  $-q$  is at rest in frame  $C$ , its electric field is just the Coulomb field from electrostatics, and there is no magnetic field. The force on  $+q$  is therefore just the Coulomb force

$$\bar{\mathbf{F}} = -\frac{q^2}{4\pi\epsilon_0 d^2} \hat{\mathbf{y}} \quad (23)$$