RELATIVISTIC TRANSFORMATION OF ELECTRIC AND MAGNETIC FIELDS: AN EXAMPLE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.45.

As an example of the transformation equations for electromagnetic fields, consider the following situation. In the lab frame A, we have a charge $-q$ moving at speed v in the $+x$ direction, and a charge $+q$ moving at the same speed v but in the $-x$ direction, with $-q$ following the path $y=0$ and $+q$ following $y=+d$. Their positions are such that their closest approach occurs when they cross the y axis.

First, we can work out the fields and the force on $+q$ at this point in the lab frame. The fields produced by a moving point charge are

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0\gamma^2}\frac{1}{\left(1-\beta^2\sin^2\theta\right)^{3/2}}\frac{\hat{\mathbf{R}}}{R^2} \tag{1}$$

$$\mathbf{B} = \frac{1}{4\pi\varepsilon_0 c^2}\frac{qv\left(1-v^2/c^2\right)\sin\theta}{\left[1-\left(v^2/c^2\right)\sin^2\theta\right]^{3/2}}\frac{\hat{\phi}}{R^2} \tag{2}$$

where $\mathbf{R}$ is the vector from the moving charge to the observer and the direction of $\hat{\phi}$ is found from using the right-hand rule on the particle's velocity $\mathbf{v}$, as usual. The angle θ is the angle between $\mathbf{R}$ and $\mathbf{v}$. In our case, at the point where the charges are at their closest approach $\theta=\pi/2$ and $R=d$ so we get

$$\mathbf{E} = -\frac{q\gamma}{4\pi\varepsilon_0 d^2}\hat{\mathbf{y}} \tag{3}$$

$$\mathbf{B} = -\frac{q\gamma v}{4\pi\varepsilon_0 c^2 d^2}\hat{\mathbf{z}} \tag{4}$$

The force on $+q$ with velocity $\mathbf{v}=-v\hat{\mathbf{x}}$ can be found from the Lorentz force law:

$$\mathbf{F} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \tag{5}$$

$$= -\frac{q^2\gamma}{4\pi\varepsilon_0 d^2}\left(\hat{\mathbf{y}}+\frac{v^2}{c^2}(-\hat{\mathbf{x}}\times\hat{\mathbf{z}})\right) \tag{6}$$

$$= -\frac{q^2\gamma}{4\pi\varepsilon_0 d^2}\left(1+\frac{v^2}{c^2}\right)\hat{\mathbf{y}} \tag{7}$$

Now suppose we switch to frame B in which $+q$ is at rest. This frame is moving with velocity $-v\hat{\mathbf{x}}$ with respect to A, so the transformation equations are

$$\bar{E}_x = E_x \tag{8}$$

$$\bar{E}_y = \gamma(E_y+vB_z) \tag{9}$$

$$\bar{E}_z = \gamma(E_z-vB_y) \tag{10}$$

$$\bar{B}_x = B_x \tag{11}$$

$$\bar{B}_y = \gamma\left(B_y-\frac{v}{c^2}E_z\right) \tag{12}$$

$$\bar{B}_z = \gamma\left(B_z+\frac{v}{c^2}E_y\right) \tag{13}$$

[Note that this is a special case where the speed v of the frame B happens to be the same as the speed of the charge in the original lab frame A, so we can use the same symbol for both. In the more general case, the v in the above 6 equations would be different from the v in equations 1 and 2.]

Only E_y and B_z are non-zero, so we get

$$\bar{\mathbf{E}} = -\frac{q\gamma^2}{4\pi\varepsilon_0 d^2}\left(1+\frac{v^2}{c^2}\right)\hat{\mathbf{y}} \tag{14}$$

$$\bar{\mathbf{B}} = -\frac{2q\gamma^2 v}{4\pi\varepsilon_0 d^2 c^2}\hat{\mathbf{z}} \tag{15}$$

[The y and $\bar{y}$, and z and $\bar{z}$ axes are parallel so we can use the unit vectors from frame A or B.] The force seen in frame B (where the velocity of $+q$ is $\mathbf{v}=0$ so there is no magnetic force) is thus

$$\bar{\mathbf{F}} = q\bar{\mathbf{E}} \tag{16}$$

$$= -\frac{q^2\gamma^2}{4\pi\varepsilon_0 d^2}\left(1+\frac{v^2}{c^2}\right)\hat{\mathbf{y}} \tag{17}$$

Note that the force in frame B is larger by a factor of γ than the force in frame A. As we saw earlier, an object experiences its maximum force in the frame in which it's at rest.

Finally, let's look at things in frame C where $-q$ is at rest. This can be found from the results from frame A by transforming to a frame moving at $+v$ relative to A, so the transformation equations are equations 9 and 13 with v replaced by $-v$, giving

$$\begin{aligned}
\bar{\mathbf{E}} &= \gamma\left(E_y - vB_z\right)\hat{\mathbf{y}} && (18)\\
&= -\frac{q\gamma^2}{4\pi\varepsilon_0 d^2}\left(1-\frac{v^2}{c^2}\right)\hat{\mathbf{y}} && (19)\\
&= -\frac{q}{4\pi\varepsilon_0 d^2}\hat{\mathbf{y}} && (20)\\
\bar{\mathbf{B}} &= \gamma\left(B_z - \frac{v}{c^2}E_y\right)\hat{\mathbf{z}} && (21)\\
&= 0 && (22)
\end{aligned}$$

Since $-q$ is at rest in frame C, its electric field is just the Coulomb field from electrostatics, and there is no magnetic field. The force on $+q$ is therefore just the Coulomb force

$$\bar{\mathbf{F}} = -\frac{q^2}{4\pi\varepsilon_0 d^2}\hat{\mathbf{y}} \tag{23}$$