

RELATIVISTIC INVARIANTS INVOLVING ELECTROMAGNETIC FIELDS

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.46.

Using the transformation equations for electromagnetic fields, we can show that there are a couple of quantities that are invariant under transformations between inertial frames.

The transformation equations are

$$\bar{E}_x = E_x \quad (1)$$

$$\bar{E}_y = \gamma(E_y + vB_z) \quad (2)$$

$$\bar{E}_z = \gamma(E_z - vB_y) \quad (3)$$

$$\bar{B}_x = B_x \quad (4)$$

$$\bar{B}_y = \gamma\left(B_y - \frac{v}{c^2}E_z\right) \quad (5)$$

$$\bar{B}_z = \gamma\left(B_z + \frac{v}{c^2}E_y\right) \quad (6)$$

First, we calculate the dot product $\mathbf{E} \cdot \mathbf{B}$ and transform it:

$$\bar{\mathbf{E}} \cdot \bar{\mathbf{B}} = E_x B_x + \gamma^2 \left[(E_y + vB_z) \left(B_y - \frac{v}{c^2} E_z \right) + (E_z - vB_y) \left(B_z + \frac{v}{c^2} E_y \right) \right] \quad (7)$$

$$= E_x B_x + \gamma^2 \left[(E_y B_y + E_z B_z) \left(1 - \frac{v^2}{c^2} \right) \right] \quad (8)$$

$$= E_x B_x + E_y B_y + E_z B_z \quad (9)$$

$$= \mathbf{E} \cdot \mathbf{B} \quad (10)$$

Second, we can find another invariant:

$$\bar{E}^2 - c^2 \bar{B}^2 = E_x^2 - c^2 B_x^2 + \gamma^2 \left[(E_y + vB_z)^2 + (E_z - vB_y)^2 - c^2 \left[\left(B_y - \frac{v}{c^2} E_z \right)^2 + \left(B_z + \frac{v}{c^2} E_y \right)^2 \right] \right] \quad (11)$$

$$= E_x^2 - c^2 B_x^2 + \gamma^2 \left[(E_y^2 + E_z^2 - c^2 B_y^2 - c^2 B_z^2) \left(1 - \frac{v^2}{c^2} \right) \right] \quad (12)$$

$$= E^2 - c^2 B^2 \quad (13)$$

These invariants put constraints on the forms given electric and magnetic fields can have in different frames. For example, if $\mathbf{B} = 0$ and $\mathbf{E} \neq 0$ in one frame, then $E^2 - c^2 B^2 > 0$ in all frames, so it's impossible to find a frame in which $\mathbf{E} = 0$. The first invariant also tells us that if \mathbf{E} and \mathbf{B} are perpendicular in one frame (as they are in an electromagnetic wave, for example) then they are perpendicular in all frames.

PINGBACKS

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