RELATIVISTIC TRANSFORMATION OF ELECTROMAGNETIC WAVES; THE DOPPLER EFFECT

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.47.

Let's apply the transformation equations for electromagnetic fields to an electromagnetic wave. Suppose we have a plane EM wave with frequency ω in the lab frame. The wave is polarized in the y direction and is travelling in the x direction, so its fields are

$$\mathbf{E} = E_0 e^{i(kx-\omega t)}\hat{\mathbf{y}} \tag{1}$$

$$\mathbf{B} = \frac{E_0}{c} e^{i(kx-\omega t)}\hat{\mathbf{z}} \tag{2}$$

where

$$k = \frac{\omega}{c} \tag{3}$$

To see how this wave looks in a frame moving with velocity v in the x direction, we can use the transformation equations

$$\bar{E}_y = \gamma\left(E_y - vB_z\right) \tag{4}$$

$$\bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right) \tag{5}$$

We get

$$\bar{\mathbf{E}} = \gamma E_0\left(1-\frac{v}{c}\right)e^{i(kx-\omega t)}\hat{\mathbf{y}} \tag{6}$$

$$= E_0\sqrt{\frac{1-v/c}{1+v/c}}e^{i(kx-\omega t)}\hat{\mathbf{y}} \tag{7}$$

$$\bar{\mathbf{B}} = \gamma\frac{E_0}{c}\left(1-\frac{v}{c}\right)e^{i(kx-\omega t)}\hat{\mathbf{z}} \tag{8}$$

$$\frac{E_0}{c}\sqrt{\frac{1-v/c}{1+v/c}}e^{i(kx-\omega t)}\hat{\mathbf{z}} \tag{9}$$

To get the final forms, we used

$$\gamma\left(1-\frac{v}{c}\right) = \frac{1-v/c}{\sqrt{1-v^2/c^2}} \tag{10}$$

$$= \frac{1-v/c}{\sqrt{(1-v/c)(1+v/c)}} \tag{11}$$

$$= \sqrt{\frac{1-v/c}{1+v/c}} \tag{12}$$

The amplitude of the wave gets smaller as the speed of the frame increases, becoming zero as $v \to c$.

To express this in the moving frame's coordinates, we use the (inverse) Lorentz transformations:

$$x = \gamma(\bar{x}+v\bar{t}) \tag{13}$$

$$t = \gamma\left(\bar{t}+\frac{v\bar{x}}{c^2}\right) \tag{14}$$

giving

$$kx-\omega t = \gamma\left(k-\frac{\omega v}{c^2}\right)\bar{x}-\gamma(\omega-kv)\bar{t} \tag{15}$$

$$= \gamma\frac{\omega}{c}\left(1-\frac{v}{c}\right)\bar{x}-\gamma\omega\left(1-\frac{v}{c}\right)\bar{t} \tag{16}$$

$$= \frac{\omega}{c}\sqrt{\frac{1-v/c}{1+v/c}}\bar{x}-\omega\sqrt{\frac{1-v/c}{1+v/c}}\bar{t} \tag{17}$$

Thus in the moving frame, the frequency of the wave is

$$\bar{\omega} = \omega\sqrt{\frac{1-v/c}{1+v/c}} \tag{18}$$

and the wavelength is

$$\begin{aligned}
\bar{\lambda} &= \frac{2\pi}{\bar{k}} &(19)\\
&= \frac{2\pi c}{\bar{\omega}} &(20)\\
&= \frac{2\pi c}{\omega}\sqrt{\frac{1+v/c}{1-v/c}} &(21)\\
&= \lambda\sqrt{\frac{1+v/c}{1-v/c}} &(22)
\end{aligned}$$

That is, the wavelength gets longer, approaching infinity as $v \to c$, while the frequency gets smaller, approaching zero as $v \to c$. This is the Doppler effect for light. If $v > 0$ so that we are moving in the direction of propagation of the wave, the wavelength gets longer resulting in a red-shift. If $v < 0$ so that we are moving against the direction of propagation, the wavelength gets shorter and we have a blue-shift.

However, note that the speed of the wave is

$$\bar{c} = \frac{\bar{\lambda}\bar{\omega}}{2\pi} = c \tag{23}$$

so not surprisingly, the speed of the wave remains the same in the moving frame.

The intensity of the wave in the lab frame is given by

$$I = \frac{E_0^2}{2} c\varepsilon_0 \tag{24}$$

In the moving frame this becomes

$$\bar{I} = \frac{E_0^2}{2} c\varepsilon_0 \frac{1-v/c}{1+v/c} \tag{25}$$

so the ratio is

$$\frac{\bar{I}}{I} = \frac{1-v/c}{1+v/c} \tag{26}$$

and the intensity drops to zero as $v \to c$.

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