

RELATIVISTIC TRANSFORMATION OF ELECTROMAGNETIC WAVES; THE DOPPLER EFFECT

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.47.

Let's apply the transformation equations for electromagnetic fields to an electromagnetic wave. Suppose we have a plane EM wave with frequency ω in the lab frame. The wave is polarized in the y direction and is travelling in the x direction, so its fields are

$$(1) \quad \mathbf{E} = E_0 e^{i(kx - \omega t)} \hat{\mathbf{y}}$$

$$(2) \quad \mathbf{B} = \frac{E_0}{c} e^{i(kx - \omega t)} \hat{\mathbf{z}}$$

where

$$(3) \quad k = \frac{\omega}{c}$$

To see how this wave looks in a frame moving with velocity v in the x direction, we can use the transformation equations

$$(4) \quad \bar{E}_y = \gamma(E_y - vB_z)$$

$$(5) \quad \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

We get

$$(6) \quad \bar{\mathbf{E}} = \gamma E_0 \left(1 - \frac{v}{c}\right) e^{i(kx - \omega t)} \hat{\mathbf{y}}$$

$$(7) \quad = E_0 \sqrt{\frac{1 - v/c}{1 + v/c}} e^{i(kx - \omega t)} \hat{\mathbf{y}}$$

$$(8) \quad \bar{\mathbf{B}} = \gamma \frac{E_0}{c} \left(1 - \frac{v}{c}\right) e^{i(kx - \omega t)} \hat{\mathbf{z}}$$

$$(9) \quad \frac{E_0}{c} \sqrt{\frac{1 - v/c}{1 + v/c}} e^{i(kx - \omega t)} \hat{\mathbf{z}}$$

To get the final forms, we used

$$(10) \quad \gamma\left(1 - \frac{v}{c}\right) = \frac{1 - v/c}{\sqrt{1 - v^2/c^2}}$$

$$(11) \quad = \frac{1 - v/c}{\sqrt{(1 - v/c)(1 + v/c)}}$$

$$(12) \quad = \sqrt{\frac{1 - v/c}{1 + v/c}}$$

The amplitude of the wave gets smaller as the speed of the frame increases, becoming zero as $v \rightarrow c$.

To express this in the moving frame's coordinates, we use the (inverse) Lorentz transformations:

$$(13) \quad x = \gamma(\bar{x} + v\bar{t})$$

$$(14) \quad t = \gamma\left(\bar{t} + \frac{v\bar{x}}{c^2}\right)$$

giving

$$(15) \quad kx - \omega t = \gamma\left(k - \frac{\omega v}{c^2}\right)\bar{x} - \gamma(\omega - kv)\bar{t}$$

$$(16) \quad = \gamma\frac{\omega}{c}\left(1 - \frac{v}{c}\right)\bar{x} - \gamma\omega\left(1 - \frac{v}{c}\right)\bar{t}$$

$$(17) \quad = \frac{\omega}{c}\sqrt{\frac{1 - v/c}{1 + v/c}}\bar{x} - \omega\sqrt{\frac{1 - v/c}{1 + v/c}}\bar{t}$$

Thus in the moving frame, the frequency of the wave is

$$(18) \quad \bar{\omega} = \omega\sqrt{\frac{1 - v/c}{1 + v/c}}$$

and the wavelength is

$$(19) \quad \bar{\lambda} = \frac{2\pi}{\bar{k}}$$

$$(20) \quad = \frac{2\pi c}{\bar{\omega}}$$

$$(21) \quad = \frac{2\pi c}{\omega} \sqrt{\frac{1+v/c}{1-v/c}}$$

$$(22) \quad = \lambda \sqrt{\frac{1+v/c}{1-v/c}}$$

That is, the wavelength gets longer, approaching infinity as $v \rightarrow c$, while the frequency gets smaller, approaching zero as $v \rightarrow c$. This is the Doppler effect for light. If $v > 0$ so that we are moving in the direction of propagation of the wave, the wavelength gets longer resulting in a red-shift. If $v < 0$ so that we are moving against the direction of propagation, the wavelength gets shorter and we have a blue-shift.

However, note that the speed of the wave is

$$(23) \quad \bar{c} = \frac{\bar{\lambda} \bar{\omega}}{2\pi} = c$$

so not surprisingly, the speed of the wave remains the same in the moving frame.

The intensity of the wave in the lab frame is given by

$$(24) \quad I = \frac{E_0^2}{2} c \epsilon_0$$

In the moving frame this becomes

$$(25) \quad \bar{I} = \frac{E_0^2}{2} c \epsilon_0 \frac{1-v/c}{1+v/c}$$

so the ratio is

$$(26) \quad \frac{\bar{I}}{I} = \frac{1-v/c}{1+v/c}$$

and the intensity drops to zero as $v \rightarrow c$.

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