SYMMETRY OF A RANK 2 TENSOR IS PRESERVED UNDER LORENTZ TRANSFORMATION

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Reference: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.49.

We've seen that the electric and magnetic fields can be represented as components of an anti-symmetric rank 2 tensor. In fact, any symmetric or anti-symmetric tensor retains its symmetry property under Lorentz transformation. We can show this by some index juggling as usual in relativity.

First, suppose the tensor T is symmetric so that $T^{ij}=T^{ji}$. Then under Lorentz transformation, we have

$$\bar{T}^{ij} = \Lambda_k^i \Lambda_l^j T^{kl} \tag{0.1}$$

$$= \Lambda_k^i \Lambda_l^j T^{lk} \tag{0.2}$$

$$= \Lambda_l^j \Lambda_k^i T^{lk} \tag{0.3}$$

$$= \bar{T}^{ji} \tag{0.4}$$

If T is anti-symmetric, then $T^{ij}=-T^{ji}$ and

$$\bar{T}^{ij} = \Lambda_k^i \Lambda_l^j T^{kl} \tag{0.5}$$

$$= -\Lambda_k^i \Lambda_l^j T^{lk} \tag{0.6}$$

$$= -\Lambda_l^j \Lambda_k^i T^{lk} \tag{0.7}$$

$$= -\bar{T}^{ji} \tag{0.8}$$

If you want to use full matrix notation, you can write

$$\bar{T} = \Lambda T \Lambda^T \tag{0.9}$$

Taking the transpose of a matrix product reverses the order of terms and takes the transpose of each term, so

$$\bar{T}^T = \left(\Lambda T \Lambda^T\right)^T \tag{0.10}$$

$$= \Lambda T^T \Lambda^T \tag{0.11}$$

[Sorry about using a T for the matrix and a superscript T for transpose, but hopefully you can keep them separate.]

Now for a symmetric matrix $T^T = T$, so $\bar{T}^T = \bar{T}$ and for an anti-symmetric matrix $T^T = -T$, so $\bar{T}^T = -\bar{T}$, showing that the symmetry property is preserved. In fact, this latter proof shows that the symmetry or anti-symmetry of T is preserved no matter what matrix Λ is.