SYMMETRY OF A RANK 2 TENSOR IS PRESERVED UNDER LORENTZ TRANSFORMATION

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Reference: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.49.

We've seen that the electric and magnetic fields can be represented as components of an anti-symmetric rank 2 tensor. In fact, any symmetric or anti-symmetric tensor retains its symmetry property under Lorentz transformation. We can show this by some index juggling as usual in relativity.

First, suppose the tensor T is symmetric so that $T^{ij}=T^{ji}$. Then under Lorentz transformation, we have

$$\begin{aligned}
\bar{T}^{ij} &= \Lambda_{k}^{i}\Lambda_{l}^{j}T^{kl} &(1)\\
&= \Lambda_{k}^{i}\Lambda_{l}^{j}T^{lk} &(2)\\
&= \Lambda_{l}^{j}\Lambda_{k}^{i}T^{lk} &(3)\\
&= \bar{T}^{ji} &(4)
\end{aligned}$$

If T is anti-symmetric, then $T^{ij}=-T^{ji}$ and

$$\begin{aligned}
\bar{T}^{ij} &= \Lambda_{k}^{i}\Lambda_{l}^{j}T^{kl} &(5)\\
&= -\Lambda_{k}^{i}\Lambda_{l}^{j}T^{lk} &(6)\\
&= -\Lambda_{l}^{j}\Lambda_{k}^{i}T^{lk} &(7)\\
&= -\bar{T}^{ji} &(8)
\end{aligned}$$

If you want to use full matrix notation, you can write

$$\bar{T}=\Lambda T\Lambda^{T} \tag{9}$$

Taking the transpose of a matrix product reverses the order of terms and takes the transpose of each term, so

$$\begin{aligned}
\bar{T}^{T} &= \left(\Lambda T\Lambda^{T}\right)^{T} &(10)\\
&= \Lambda T^{T}\Lambda^{T} &(11)
\end{aligned}$$

[Sorry about using a T for the matrix and a superscript T for transpose, but hopefully you can keep them separate.]

Now for a symmetric matrix $T^T = T$, so $\bar{T}^T = \bar{T}$ and for an anti-symmetric matrix $T^T = -T$, so $\bar{T}^T = -\bar{T}$, showing that the symmetry property is preserved. In fact, this latter proof shows that the symmetry or anti-symmetry of T is preserved no matter what matrix Λ is.