SYMMETRY OF A RANK 2 TENSOR IS PRESERVED UNDER LORENTZ TRANSFORMATION

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Reference: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.49.

We've seen that the electric and magnetic fields can be represented as components of an anti-symmetric rank 2 tensor. In fact, any symmetric or anti-symmetric tensor retains its symmetry property under Lorentz transformation. We can show this by some index juggling as usual in relativity.

First, suppose the tensor T is symmetric so that $T^{ij} = T^{ji}$. Then under Lorentz transformation, we have

$$
\begin{aligned}
\bar{T}^{ij} &= \Lambda^i_k \Lambda^j_l T^{kl} & (1)\\
&= \Lambda^i_k \Lambda^j_l T^{lk} & (2)\\
&= \Lambda^j_l \Lambda^i_k T^{lk} & (3)\\
&= \bar{T}^{ji} & (4)
\end{aligned}
$$

If T is anti-symmetric, then $T^{ij} = -T^{ji}$ and

$$
\begin{aligned}
\bar{T}^{ij} &= \Lambda^i_k \Lambda^j_l T^{kl} & (5)\\
&= -\Lambda^i_k \Lambda^j_l T^{lk} & (6)\\
&= -\Lambda^j_l \Lambda^i_k T^{lk} & (7)\\
&= -\bar{T}^{ji} & (8)
\end{aligned}
$$

If you want to use full matrix notation, you can write

$$
\bar{T} = \Lambda T \Lambda^T \tag{9}
$$

Taking the transpose of a matrix product reverses the order of terms and takes the transpose of each term, so

$$
\begin{aligned}
\bar{T}^T &= \left(\Lambda T \Lambda^T\right)^T & (10)\\
&= \Lambda T^T \Lambda^T & (11)
\end{aligned}
$$

[Sorry about using a T for the matrix and a superscript T for transpose, but hopefully you can keep them separate.]

Now for a symmetric matrix $T^T = T$, so $\bar{T}^T = \bar{T}$ and for an anti-symmetric matrix $T^T = -T$, so $\bar{T}^T = -\bar{T}$, showing that the symmetry property is preserved. In fact, this latter proof shows that the symmetry or anti-symmetry of T is preserved no matter what matrix Λ is.