THE DUAL ELECTROMAGNETIC FIELD TENSOR

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Reference: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problems 12.50-51.

Here are a couple of examples involving the electromagnetic field tensor

$$F^{ij}=\begin{bmatrix}0 & E_x & E_y & E_z\\ -E_x & 0 & B_z & -B_y\\ -E_y & -B_z & 0 & B_x\\ -E_z & B_y & -B_x & 0\end{bmatrix} \tag{1}$$

The tensor leads to the Lorentz transformations of the fields:

$$E'_x = E_x \tag{2}$$
$$E'_y = \gamma E_y-\gamma\beta B_z \tag{3}$$
$$E'_z = \gamma E_z+\gamma\beta B_y \tag{4}$$
$$B'_x = B_x \tag{5}$$
$$B'_y = \gamma\beta E_z+\gamma B_y \tag{6}$$
$$B'_z = -\gamma\beta E_y+\gamma B_z \tag{7}$$

Actually, the same transformations can be obtained by replacing **E** by **B**, and **B** by $-\mathbf{E}$ (using $c=1$) in the original tensor. This gives us another rank-2 tensor which is the dual tensor to F^{ij}:

$$G^{ij}=\begin{bmatrix}0 & B_x & B_y & B_z\\ -B_x & 0 & -E_z & E_y\\ -B_y & E_z & 0 & -E_x\\ -B_z & -E_y & E_x & 0\end{bmatrix} \tag{8}$$

If we lower both indices in these two tensors, we change the signs of all elements in the first row and first column, to get

$$F_{ij} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix} \tag{9}$$

$$G_{ij} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z & E_y \\ B_y & E_z & 0 & -E_x \\ B_z & -E_y & E_x & 0 \end{bmatrix} \tag{10}$$

We can now calculate some invariants by finding

$$F^{ij}F_{ij} = -2E^2 + 2B^2 \tag{11}$$

$$G^{ij}G_{ij} = -2B^2 + 2E^2 \tag{12}$$

$$F^{ij}G_{ij} = -2\mathbf{E}\cdot\mathbf{B} - 2\mathbf{E}\cdot\mathbf{B} = -4\mathbf{E}\cdot\mathbf{B} \tag{13}$$

Comparing these results with those got earlier by directly calculating the Lorentz transformation of these quantities, we see that the tensor products give the same invariants (after restoring the factors of c).

As simple of example of calculating the elements in the tensors, suppose we have an infinite straight wire along the z axis with linear charge density λ moving at speed v. From Gauss's law the electric field a distance x from the wire is

$$\mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0 x}\hat{\mathbf{r}} \tag{14}$$

and from Ampère's law the magnetic field is

$$\mathbf{B} = \frac{\mu_0 \lambda v}{2\pi x}\hat{\phi} \tag{15}$$

so at point $(x,0,0)$ we have

$$E_x = \frac{\lambda}{2\pi\varepsilon_0 x} \tag{16}$$

$$E_y = E_z = 0 \tag{17}$$

$$B_y = \frac{\mu_0 \lambda v}{2\pi x} \tag{18}$$

$$B_x = B_z = 0 \tag{19}$$

so the tensors are (again with $c=1$)

$$F^{ij} = \begin{bmatrix} 0 & \frac{\lambda}{2\pi\varepsilon_0 x} & 0 & 0 \\ -\frac{\lambda}{2\pi\varepsilon_0 x} & 0 & 0 & -\frac{\mu_0\lambda v}{2\pi x} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\mu_0\lambda v}{2\pi x} & 0 & 0 \end{bmatrix} \tag{20}$$

$$G^{ij} = \begin{bmatrix} 0 & 0 & \frac{\mu_0\lambda v}{2\pi x} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\mu_0\lambda v}{2\pi x} & 0 & 0 & -\frac{\lambda}{2\pi\varepsilon_0 x} \\ 0 & 0 & \frac{\lambda}{2\pi\varepsilon_0 x} & 0 \end{bmatrix} \tag{21}$$