

## ELECTROMAGNETIC FIELD TENSOR: A COUPLE OF EXAMPLES

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Reference: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problems 12.52-53.

Maxwell's equations can be expressed quite simply in terms of the electromagnetic field tensor

$$F^{ij} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix} \quad (1)$$

and its dual tensor:

$$G^{ij} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{bmatrix} \quad (2)$$

In his section 12.3.4, Griffiths shows that all four of Maxwell's equations can be expressed as

$$\partial_j F^{ij} = \mu_0 J^i \quad (3)$$

$$\partial_j G^{ij} = 0 \quad (4)$$

where  $J^i$  is a four-vector known as the four-current with components

$$J^i = [c\rho, J_x, J_y, J_z] \quad (5)$$

$$= \frac{\rho_0}{\sqrt{1 - u^2/c^2}} [c, u_x, u_y, u_z] \quad (6)$$

In this formula, we're looking at a small element of charge that is moving with velocity  $\mathbf{u}$  and has charge density  $\rho_0$  in its rest frame. The four-current is just  $\rho_0$  times the four-velocity:

$$J^i = \rho_0 \eta^i \quad (7)$$

If we take the divergence (in four dimensions) of 3 we get, since  $F^{ij} = -F^{ji}$

$$\partial_i \partial_j F^{ij} = -\partial_i \partial_j F^{ji} = 0 \quad (8)$$

which leads to

$$\partial_i J^i = 0 \quad (9)$$

This is a compact way of expressing the continuity condition, since from 5

$$\partial_i J^i = \frac{\partial (c\rho)}{\partial x^0} + \nabla \cdot \mathbf{J} \quad (10)$$

$$= \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \quad (11)$$

The equation 4 can be written in terms of  $F_{ij}$  as follows. First, we can write  $G^{ij}$  in terms of the components of  $F_{ij}$  by comparing the two matrices 2 and  $F_{ij}$ :

$$F_{ij} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix} \quad (12)$$

We get

$$G^{ij} = \begin{bmatrix} 0 & F_{23} & F_{31} & F_{12} \\ F_{32} & 0 & F_{03} & F_{20} \\ F_{13} & F_{30} & 0 & F_{01} \\ F_{21} & F_{02} & F_{10} & 0 \end{bmatrix} \quad (13)$$

Now we can read off the four equations contained in 4 by scanning the rows of this matrix:

$$\partial_j G^{0j} = \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} \quad (14)$$

$$\partial_j G^{1j} = \partial_0 F_{32} + \partial_2 F_{03} + \partial_3 F_{20} \quad (15)$$

$$\partial_j G^{2j} = \partial_0 F_{13} + \partial_1 F_{30} + \partial_3 F_{01} \quad (16)$$

$$\partial_j G^{3j} = \partial_0 F_{21} + \partial_1 F_{02} + \partial_2 F_{10} \quad (17)$$

All four of these equations have the form

$$\partial_j G^{ij} = \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} \quad (18)$$

where  $i$ ,  $a$ ,  $b$  and  $c$  are all different. Note that the indexes  $a$ ,  $b$  and  $c$  are cyclically permuted in each term on the RHS. Therefore we can write 4 as

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0 \quad (19)$$

where  $a$ ,  $b$  and  $c$  are a subset of  $\{0, 1, 2, 3\}$ .

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