ELECTROMAGNETIC FIELD TENSOR: A COUPLE OF EXAMPLES

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Reference: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problems 12.52-53.

Maxwell's equations can be expressed quite simply in terms of the electromagnetic field tensor

$$F^{ij}=\left[\begin{array}{cccc}0 & E_x & E_y & E_z\\ -E_x & 0 & B_z & -B_y\\ -E_y & -B_z & 0 & B_x\\ -E_z & B_y & -B_x & 0\end{array}\right] \tag{1}$$

and its dual tensor:

$$G^{ij}=\left[\begin{array}{cccc}0 & B_x & B_y & B_z\\ -B_x & 0 & -E_z & E_y\\ -B_y & E_z & 0 & -E_x\\ -B_z & -E_y & E_x & 0\end{array}\right] \tag{2}$$

In his section 12.3.4, Griffiths shows that all four of Maxwell's equations can be expressed as

$$\begin{aligned}\partial_j F^{ij} &= \mu_0 J^i && (3)\\ \partial_j G^{ij} &= 0 && (4)\end{aligned}$$

where J^i is a four-vector known as the four-current with components

$$\begin{aligned}J^i &= [\,c\rho, J_x, J_y, J_z] && (5)\\ &= \frac{\rho_0}{\sqrt{1-u^2/c^2}}[\,c, u_x, u_y, u_z] && (6)\end{aligned}$$

In this formula, we're looking at a small element of charge that is moving with velocity **u** and has charge density ρ_0 in its rest frame. The four-current is just ρ_0 times the four-velocity:

$$J^i=\rho_0\eta^i \tag{7}$$

If we take the divergence (in four dimensions) of 3 we get, since $F^{ij} = -F^{ji}$

$$\partial_i \partial_j F^{ij} = -\partial_i \partial_j F^{ji} = 0 \tag{8}$$

which leads to

$$\partial_i J^i = 0 \tag{9}$$

This is a compact way of expressing the continuity condition, since from 5

$$\begin{aligned} \partial_i J^i &= \frac{\partial (c\rho)}{\partial x^0} + \nabla \cdot \mathbf{J} \qquad (10) \\ &= \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \qquad (11) \end{aligned}$$

The equation 4 can be written in terms of F_{ij} as follows. First, we can write G^{ij} in terms of the components of F_{ij} by comparing the two matrices 2 and F_{ij}:

$$F_{ij} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix} \tag{12}$$

We get

$$G^{ij} = \begin{bmatrix} 0 & F_{23} & F_{31} & F_{12} \\ F_{32} & 0 & F_{03} & F_{20} \\ F_{13} & F_{30} & 0 & F_{01} \\ F_{21} & F_{02} & F_{10} & 0 \end{bmatrix} \tag{13}$$

Now we can read off the four equations contained in 4 by scanning the rows of this matrix:

$$\begin{aligned} \partial_j G^{0j} &= \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} \qquad (14) \\ \partial_j G^{1j} &= \partial_0 F_{32} + \partial_2 F_{03} + \partial_3 F_{20} \qquad (15) \\ \partial_j G^{2j} &= \partial_0 F_{13} + \partial_1 F_{30} + \partial_3 F_{01} \qquad (16) \\ \partial_j G^{3j} &= \partial_0 F_{21} + \partial_1 F_{02} + \partial_2 F_{10} \qquad (17) \end{aligned}$$

All four of these equations have the form

$$\partial_j G^{ij} = \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} \tag{18}$$

where i, a, b and c are all different. Note that the indexes a, b and c are cyclically permuted in each term on the RHS. Therefore we can write 4 as

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0 \tag{19}$$

where a, b and c are a subset of $\{0,1,2,3\}$.

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