ELECTROMAGNETIC FIELD TENSOR: A COUPLE OF EXAMPLES

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Reference: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problems 12.52-53.

Maxwell's equations can be expressed quite simply in terms of the electromagnetic field tensor

$$F^{ij}=\left[\begin{array}{cccc}0 & E_x & E_y & E_z\\ -E_x & 0 & B_z & -B_y\\ -E_y & -B_z & 0 & B_x\\ -E_z & B_y & -B_x & 0\end{array}\right] \tag{0.1}$$

and its dual tensor:

$$G^{ij}=\left[\begin{array}{cccc}0 & B_x & B_y & B_z\\ -B_x & 0 & -E_z & E_y\\ -B_y & E_z & 0 & -E_x\\ -B_z & -E_y & E_x & 0\end{array}\right] \tag{0.2}$$

In his section 12.3.4, Griffiths shows that all four of Maxwell's equations can be expressed as

$$\partial_j F^{ij} = \mu_0 J^i \tag{0.3}$$

$$\partial_j G^{ij} = 0 \tag{0.4}$$

where J^i is a four-vector known as the four-current with components

$$J^i = [\,c\rho, J_x, J_y, J_z] \tag{0.5}$$

$$= \frac{\rho_0}{\sqrt{1-u^2/c^2}}[\,c, u_x, u_y, u_z] \tag{0.6}$$

In this formula, we're looking at a small element of charge that is moving with velocity $\mathbf{u}$ and has charge density ρ_0 in its rest frame. The four-current is just ρ_0 times the four-velocity:

$$J^i = \rho_0 \eta^i \tag{0.7}$$

If we take the divergence (in four dimensions) of 0.3 we get, since $F^{ij} = -F^{ji}$

$$\partial_i \partial_j F^{ij} = -\partial_i \partial_j F^{ji} = 0 \tag{0.8}$$

which leads to

$$\partial_i J^i = 0 \tag{0.9}$$

This is a compact way of expressing the continuity condition, since from 0.5

$$\partial_i J^i = \frac{\partial (c\rho)}{\partial x^0} + \nabla \cdot \mathbf{J} \tag{0.10}$$

$$= \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{0.11}$$

The equation 0.4 can be written in terms of F_{ij} as follows. First, we can write G^{ij} in terms of the components of F_{ij} by comparing the two matrices 0.2 and F_{ij}:

$$F_{ij} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix} \tag{0.12}$$

We get

$$G^{ij} = \begin{bmatrix} 0 & F_{23} & F_{31} & F_{12} \\ F_{32} & 0 & F_{03} & F_{20} \\ F_{13} & F_{30} & 0 & F_{01} \\ F_{21} & F_{02} & F_{10} & 0 \end{bmatrix} \tag{0.13}$$

Now we can read off the four equations contained in 0.4 by scanning the rows of this matrix:

$$\partial_j G^{0j} = \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} \tag{0.14}$$
$$\partial_j G^{1j} = \partial_0 F_{32} + \partial_2 F_{03} + \partial_3 F_{20} \tag{0.15}$$
$$\partial_j G^{2j} = \partial_0 F_{13} + \partial_1 F_{30} + \partial_3 F_{01} \tag{0.16}$$
$$\partial_j G^{3j} = \partial_0 F_{21} + \partial_1 F_{02} + \partial_2 F_{10} \tag{0.17}$$

All four of these equations have the form

$$\partial_j G^{ij} = \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} \tag{0.18}$$

where i, a, b and c are all different. Note that the indexes a, b and c are cyclically permuted in each term on the RHS. Therefore we can write 0.4 as

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0 \tag{0.19}$$

where a, b and c are a subset of $\{0, 1, 2, 3\}$.

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