

## CONTRAVARIANT GRADIENT OPERATOR

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Reference: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.55.

The gradient of a scalar function  $\phi$  is a covariant vector since it transforms as

$$\frac{\partial \phi}{\partial \bar{x}^a} = \frac{\partial \phi}{\partial x^i} \frac{\partial x^i}{\partial \bar{x}^a} \quad (1)$$

We can therefore regard the gradient operator  $\partial_a$  on its own as a covariant vector, so it should have a contravariant counterpart. In flat space, the only change in switching from covariant to contravariant is that the time component changes sign. Given that the Lorentz transformation for a contravariant four-vector is

$$\bar{x}^0 = \gamma(x^0 - \beta x^1) \quad (2)$$

$$\bar{x}^1 = \gamma(x^1 - \beta x^0) \quad (3)$$

$$\bar{x}^2 = x^2 \quad (4)$$

$$\bar{x}^3 = x^3 \quad (5)$$

the transformations for the covariant four-vector are obtained by lowering all indices and replacing the time components by their negatives:

$$\bar{x}_0 = \gamma(x_0 + \beta x_1) \quad (6)$$

$$\bar{x}_1 = \gamma(x_1 + \beta x_0) \quad (7)$$

$$\bar{x}_2 = x_2 \quad (8)$$

$$\bar{x}_3 = x_3 \quad (9)$$

where we multiplied the  $\bar{x}_0$  equation through by  $-1$ . The corresponding inverse transformations are obtained by replacing  $\beta$  by  $-\beta$ :

$$x_0 = \gamma(\bar{x}_0 - \beta\bar{x}_1) \quad (10)$$

$$x_1 = \gamma(\bar{x}_1 - \beta\bar{x}_0) \quad (11)$$

$$x_2 = \bar{x}_2 \quad (12)$$

$$x_3 = \bar{x}_3 \quad (13)$$

Thus the *inverse* covariant transformations are the same as the *forward* contravariant transformations.

The contravariant gradient is  $\partial^i\phi = \frac{\partial\phi}{\partial x_i}$  so

$$\overline{\partial^i\phi} = \frac{\partial\phi}{\partial \bar{x}_i} \quad (14)$$

$$= \frac{\partial\phi}{\partial x_k} \frac{\partial x_k}{\partial \bar{x}_i} \quad (15)$$

$$= \partial^k\phi \frac{\partial x_k}{\partial \bar{x}_i} \quad (16)$$

The transformations for each value of  $i$  are then

$$\overline{\partial^0\phi} = \gamma\partial^0\phi - \beta\gamma\partial^1\phi \quad (17)$$

$$\overline{\partial^1\phi} = \gamma\partial^1\phi - \beta\gamma\partial^0\phi \quad (18)$$

$$\overline{\partial^2\phi} = \partial^2\phi \quad (19)$$

$$\overline{\partial^3\phi} = \partial^3\phi \quad (20)$$

Thus  $\partial^i\phi$  transforms like a contravariant vector.

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