CONTRAVARIANT GRADIENT OPERATOR

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Reference: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.55.

The gradient of a scalar function ϕ is a covariant vector since it transforms as

$$\frac{\partial\phi}{\partial\bar{x}^a}=\frac{\partial\phi}{\partial x^i}\frac{\partial x^i}{\partial\bar{x}^a} \tag{1}$$

We can therefore regard the gradient operator ∂_a on its own as a covariant vector, so it should have a contravariant counterpart. In flat space, the only change in switching from covariant to contravariant is that the time component changes sign. Given that the Lorentz transformation for a contravariant four-vector is

$$\bar{x}^0 = \gamma\left(x^0-\beta x^1\right) \tag{2}$$
$$\bar{x}^1 = \gamma\left(x^1-\beta x^0\right) \tag{3}$$
$$\bar{x}^2 = x^2 \tag{4}$$
$$\bar{x}^3 = x^3 \tag{5}$$

the transformations for the covariant four-vector are obtained by lowering all indices and replacing the time components by their negatives:

$$\bar{x}_0 = \gamma\left(x_0+\beta x_1\right) \tag{6}$$
$$\bar{x}_1 = \gamma\left(x_1+\beta x_0\right) \tag{7}$$
$$\bar{x}_2 = x_2 \tag{8}$$
$$\bar{x}_3 = x_3 \tag{9}$$

where we multiplied the $\bar{x}_0$ equation through by -1. The corresponding inverse transformations are obtained by replacing β by $-\beta$:

$$x_0 = \gamma(\bar{x}_0 - \beta\bar{x}_1) \tag{10}$$
$$x_1 = \gamma(\bar{x}_1 - \beta\bar{x}_0) \tag{11}$$
$$x_2 = \bar{x}_2 \tag{12}$$
$$x_3 = \bar{x}_3 \tag{13}$$

Thus the *inverse* covariant transformations are the same as the *forward* contravariant transformations.

The contravariant gradient is $\partial^i\phi = \frac{\partial\phi}{\partial x_i}$ so

$$\overline{\partial^i\phi} = \frac{\partial\phi}{\partial\bar{x}_i} \tag{14}$$
$$= \frac{\partial\phi}{\partial x_k}\frac{\partial x_k}{\partial\bar{x}_i} \tag{15}$$
$$= \partial^k\phi\frac{\partial x_k}{\partial\bar{x}_i} \tag{16}$$

The transformations for each value of i are then

$$\overline{\partial^0\phi} = \gamma\partial^0\phi - \beta\gamma\partial^1\phi \tag{17}$$
$$\overline{\partial^1\phi} = \gamma\partial^1\phi - \beta\gamma\partial^0\phi \tag{18}$$
$$\overline{\partial^2\phi} = \partial^2\phi \tag{19}$$
$$\overline{\partial^3\phi} = \partial^3\phi \tag{20}$$

Thus $\partial^i\phi$ transforms like a contravariant vector.

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