CONTRAVARIANT GRADIENT OPERATOR

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Reference: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.55.

The gradient of a scalar function ϕ is a covariant vector since it transforms as

$$\frac{\partial\phi}{\partial\bar{x}^{a}}=\frac{\partial\phi}{\partial x^{i}}\frac{\partial x^{i}}{\partial\bar{x}^{a}} \tag{1}$$

We can therefore regard the gradient operator ∂_a on its own as a covariant vector, so it should have a contravariant counterpart. In flat space, the only change in switching from covariant to contravariant is that the time component changes sign. Given that the Lorentz transformation for a contravariant four-vector is

$$\bar{x}^{0} = \gamma\left(x^{0}-\beta x^{1}\right) \tag{2}$$
$$\bar{x}^{1} = \gamma\left(x^{1}-\beta x^{0}\right) \tag{3}$$
$$\bar{x}^{2} = x^{2} \tag{4}$$
$$\bar{x}^{3} = x^{3} \tag{5}$$

the transformations for the covariant four-vector are obtained by lowering all indices and replacing the time components by their negatives:

$$\bar{x}_{0} = \gamma\left(x_{0}+\beta x_{1}\right) \tag{6}$$
$$\bar{x}_{1} = \gamma\left(x_{1}+\beta x_{0}\right) \tag{7}$$
$$\bar{x}_{2} = x_{2} \tag{8}$$
$$\bar{x}_{3} = x_{3} \tag{9}$$

where we multiplied the $\bar{x}_0$ equation through by -1. The corresponding inverse transformations are obtained by replacing β by $-\beta$:

$$
\begin{aligned}
x_0 &= \gamma(\bar{x}_0 - \beta\bar{x}_1) && (10)\\
x_1 &= \gamma(\bar{x}_1 - \beta\bar{x}_0) && (11)\\
x_2 &= \bar{x}_2 && (12)\\
x_3 &= \bar{x}_3 && (13)
\end{aligned}
$$

Thus the *inverse* covariant transformations are the same as the *forward* contravariant transformations.

The contravariant gradient is $\partial^i\phi = \frac{\partial\phi}{\partial x_i}$ so

$$
\begin{aligned}
\overline{\partial^i\phi} &= \frac{\partial\phi}{\partial\bar{x}_i} && (14)\\
&= \frac{\partial\phi}{\partial x_k}\frac{\partial x_k}{\partial\bar{x}_i} && (15)\\
&= \partial^k\phi\frac{\partial x_k}{\partial\bar{x}_i} && (16)
\end{aligned}
$$

The transformations for each value of i are then

$$
\begin{aligned}
\overline{\partial^0\phi} &= \gamma\partial^0\phi - \beta\gamma\partial^1\phi && (17)\\
\overline{\partial^1\phi} &= \gamma\partial^1\phi - \beta\gamma\partial^0\phi && (18)\\
\overline{\partial^2\phi} &= \partial^2\phi && (19)\\
\overline{\partial^3\phi} &= \partial^3\phi && (20)
\end{aligned}
$$

Thus $\partial^i\phi$ transforms like a contravariant vector.

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