RELATIVISTIC ELECTROMAGNETIC POTENTIALS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.56.

Maxwell's equations can be written in terms of the electromagnetic field tensor, its dual and the current four-vector as

$$\partial_j F^{ij} = \mu_0 J^i \tag{1}$$

$$\partial_j G^{ij} = 0 \tag{2}$$

where

$$F^{ij} = \left[\begin{array}{cccc} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{array}\right] \tag{3}$$

$$G^{ij} = \left[\begin{array}{cccc} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{array}\right] \tag{4}$$

$$J^i = [\rho, J_x, J_y, J_z] \tag{5}$$

$$= \frac{\rho_0}{\sqrt{1-\beta^2}}[1, u_x, u_y, u_z] \tag{6}$$

It turns out that an even more compact form for Maxwell's equations can be written using the 4-vector potential

$$A^i = \left(\frac{V}{c}, A_x, A_y, A_z\right) \tag{7}$$

$$= (V, A_x, A_y, A_z) \tag{8}$$

where the last line uses relativistic units where $c = 1$.

Griffiths shows in section 12.3.5 that the field tensor can be written in terms of the potentials as

$$F^{ij} = \partial^i A^j - \partial^j A^i \tag{9}$$

Note that we're using the contravariant gradient operator here, in order to get the signs right on the time components. Because of the form of F^{ij}, the gauge invariance shows up naturally, since if we replace A^i by

$$A^i \to A^i + \partial^i \lambda \tag{10}$$

where λ is any scalar function, F^{ij} is unchanged, as the order in which the partial derivatives of λ are taken doesn't matter, so λ drops out of the equation for F^{ij}. The Lorenz gauge condition is

$$\nabla \cdot \mathbf{A} = -\frac{\partial V}{\partial t} = -\partial_0 V \tag{11}$$

[Notice we're back to using the covariant gradient operator.] This can be condensed to read

$$\partial_i A^i = 0 \tag{12}$$

[Incidentally, Griffiths's equation 12.135 is wrong; it should read $\partial A^\mu / \partial x^\mu = 0$.]

Combining 1 with 9 gives

$$\partial_j \partial^i A^j - \partial_j \partial^j A^i = \mu_0 J^i \tag{13}$$

If we use the Lorenz gauge, the first term is zero, so we get

$$\partial_j \partial^j A^i = -\mu_0 J^i \tag{14}$$
$$\Box^2 A^i = -\mu_0 J^i \tag{15}$$

where the symbol $\Box^2$ is the d'Alembertian operator, defined as

$$\Box^2 \equiv \partial_j \partial^j \tag{16}$$

We can verify that the other Maxwell equation 2 is also satisfied by the potential formulation by using the earlier result

$$\partial_j G^{ij} = \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} \tag{17}$$

where i, a, b and c are all different.

Lowering the indexes on 9 we get

$$F_{ij} = \partial_i A_j - \partial_j A_i \tag{18}$$

Substituting 18 into 17 we get

$$\partial_j G^{ij} = \partial_a \partial_b A_c - \partial_a \partial_c A_b + \partial_b \partial_c A_a - \partial_b \partial_a A_c + \partial_c \partial_a A_b - \partial_c \partial_b A_a = 0 \tag{19}$$