

## RELATIVISTIC ELECTROMAGNETIC POTENTIALS

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Reference: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.56.

Maxwell's equations can be written in terms of the electromagnetic field tensor, its dual and the current four-vector as

$$(1) \quad \partial_j F^{ij} = \mu_0 J^i$$

$$(2) \quad \partial_j G^{ij} = 0$$

where

$$(3) \quad F^{ij} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix}$$

$$(4) \quad G^{ij} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{bmatrix}$$

$$(5) \quad J^i = [\rho, J_x, J_y, J_z]$$

$$(6) \quad = \frac{\rho_0}{\sqrt{1-\beta^2}} [1, u_x, u_y, u_z]$$

It turns out that an even more compact form for Maxwell's equations can be written using the 4-vector potential

$$(7) \quad A^i = \left( \frac{V}{c}, A_x, A_y, A_z \right)$$

$$(8) \quad = (V, A_x, A_y, A_z)$$

where the last line uses relativistic units where  $c = 1$ .

Griffiths shows in section 12.3.5 that the field tensor can be written in terms of the potentials as

$$(9) \quad F^{ij} = \partial^i A^j - \partial^j A^i$$

Note that we're using the contravariant gradient operator here, in order to get the signs right on the time components. Because of the form of  $F^{ij}$ , the gauge invariance shows up naturally, since if we replace  $A^i$  by

$$(10) \quad A^i \rightarrow A^i + \partial^i \lambda$$

where  $\lambda$  is any scalar function,  $F^{ij}$  is unchanged, as the order in which the partial derivatives of  $\lambda$  are taken doesn't matter, so  $\lambda$  drops out of the equation for  $F^{ij}$ . The Lorenz gauge condition is

$$(11) \quad \nabla \cdot \mathbf{A} = -\frac{\partial V}{\partial t} = -\partial_0 V$$

[Notice we're back to using the covariant gradient operator.] This can be condensed to read

$$(12) \quad \partial_i A^i = 0$$

[Incidentally, Griffiths's equation 12.135 is wrong; it should read  $\partial A^\mu / \partial x^\mu = 0$ .]

Combining 1 with 9 gives

$$(13) \quad \partial_j \partial^i A^j - \partial_j \partial^j A^i = \mu_0 J^i$$

If we use the Lorenz gauge, the first term is zero, so we get

$$(14) \quad \partial_j \partial^j A^i = -\mu_0 J^i$$

$$(15) \quad \square^2 A^i = -\mu_0 J^i$$

where the symbol  $\square^2$  is the d'Alembertian operator, defined as

$$(16) \quad \square^2 \equiv \partial_j \partial^j$$

We can verify that the other Maxwell equation 2 is also satisfied by the potential formulation by using the earlier result

$$(17) \quad \partial_j G^{ij} = \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab}$$

where  $i, a, b$  and  $c$  are all different.

Lowering the indexes on 9 we get

$$(18) \quad F_{ij} = \partial_i A_j - \partial_j A_i$$

Substituting 18 into 17 we get

$$(19) \quad \partial_j G^{ij} = \partial_a \partial_b A_c - \partial_a \partial_c A_b + \partial_b \partial_c A_a - \partial_b \partial_a A_c + \partial_c \partial_a A_b - \partial_c \partial_b A_a = 0$$