# LORENTZ TRANSFORMATION IN TWO DIMENSIONS

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Reference: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.57.

Although we've looked at the Lorentz transformations for a general 3-d motion of one frame relative to another, we'll have a look here at the slightly more specialized case of general 2-d motion.

Suppose that frame $\bar{\mathcal{S}}$ moves relative to $\mathcal{S}$ with velocity

$$\mathbf{v}=\beta c\left(\cos\phi\hat{\mathbf{x}}+\sin\phi\hat{\mathbf{y}}\right) \tag{1}$$

that is, its direction makes an angle $\phi$ with the $x$ axis. To get the Lorentz transformation here, we use the fact that distances perpendicular to $\mathbf{v}$ are unaffected, and distances parallel to $\mathbf{v}$ transform using the regular 1-d Lorentz transformation. It's therefore easiest to transform to a lab frame $\mathcal{S}'$ that is rotated relative to $\mathcal{S}$ by $\phi$, so that the basis vectors are

$$\hat{\mathbf{x}}' = \cos\phi\hat{\mathbf{x}}+\sin\phi\hat{\mathbf{y}} \tag{2}$$

$$\hat{\mathbf{y}}' = -\sin\phi\hat{\mathbf{x}}+\cos\phi\hat{\mathbf{y}} \tag{3}$$

A point with coordinates $[x',y']$ in $\mathcal{S}'$ therefore has coordinates in terms of $\mathcal{S}$ coordinates of

$$x' = x\cos\phi+y\sin\phi \tag{4}$$

$$y' = -x\sin\phi+y\cos\phi \tag{5}$$

In this system, the Lorentz transformation is

$$\Lambda'=\left[\begin{array}{cccc}
\gamma & -\beta\gamma & 0 & 0\\
-\beta\gamma & \gamma & 0 & 0\\
0 & 0 & 1 & 0\\
0 & 0 & 0 & 1
\end{array}\right] \tag{6}$$

We can write this out explicitly for each coordinate. The time coordinate is the same both $\mathcal{S}$ and $\mathcal{S}'$ since the two frames are at rest relative to each other, so

$$\bar{t}' = \gamma\left(t-\beta x'\right) \tag{7}$$
$$\bar{t} = \gamma t-\gamma\beta x\cos\phi-\gamma\beta y\sin\phi \tag{8}$$

The $x'$ coordinate transforms as follows:

$$\bar{x}' = \gamma\left(x'-\beta t\right) \tag{9}$$
$$\bar{x}\cos\phi+\bar{y}\sin\phi = \gamma x\cos\phi+\gamma y\sin\phi-\gamma\beta t \tag{10}$$

And $y'$ transforms as

$$\bar{y}' = y' \tag{11}$$
$$-\bar{x}\sin\phi+\bar{y}\cos\phi = -x\sin\phi+y\cos\phi \tag{12}$$

Multiplying 10 by $\cos\phi$ and 12 by $-\sin\phi$ and adding, we get

$$\bar{x} = -\beta\gamma t\cos\phi+\left(\gamma\cos^2\phi+\sin^2\phi\right)x+(\gamma-1)\,y\sin\phi\cos\phi \tag{13}$$

Multiplying 10 by $\sin\phi$ and 12 by $\cos\phi$ and adding, we get

$$\bar{y} = -\beta\gamma t\sin\phi+(\gamma-1)\,x\sin\phi\cos\phi+\left(\gamma\sin^2\phi+\cos^2\phi\right)y \tag{14}$$

Combining 8, 13 and 14 (along with $\bar{z}=z$) into a matrix, we get

$$\Lambda = \left[\begin{array}{cccc} \gamma & -\gamma\beta\cos\phi & -\gamma\beta\sin\phi & 0\\ -\beta\gamma\cos\phi & \gamma\cos^2\phi+\sin^2\phi & (\gamma-1)\sin\phi\cos\phi & 0\\ -\beta\gamma\sin\phi & (\gamma-1)\sin\phi\cos\phi & \gamma\sin^2\phi+\cos^2\phi & 0\\ 0 & 0 & 0 & 1 \end{array}\right] \tag{15}$$

This reduces to 6 when $\phi=0$, so that the axes of the two frames are parallel. This 2-d transformation matrix is also symmetric.