

LORENTZ TRANSFORMATION IN TWO DIMENSIONS

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Reference: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.57.

Although we've looked at the Lorentz transformations for a general 3-d motion of one frame relative to another, we'll have a look here at the slightly more specialized case of general 2-d motion.

Suppose that frame \mathcal{S}' moves relative to \mathcal{S} with velocity

$$(0.1) \quad \mathbf{v} = \beta c (\cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}})$$

that is, its direction makes an angle ϕ with the x axis. To get the Lorentz transformation here, we use the fact that distances perpendicular to \mathbf{v} are unaffected, and distances parallel to \mathbf{v} transform using the regular 1-d Lorentz transformation. It's therefore easiest to transform to a lab frame \mathcal{S}' that is rotated relative to \mathcal{S} by ϕ , so that the basis vectors are

$$(0.2) \quad \hat{\mathbf{x}}' = \cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}}$$

$$(0.3) \quad \hat{\mathbf{y}}' = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}}$$

A point with coordinates $[x', y']$ in \mathcal{S}' therefore has coordinates in terms of \mathcal{S} coordinates of

$$(0.4) \quad x' = x \cos \phi + y \sin \phi$$

$$(0.5) \quad y' = -x \sin \phi + y \cos \phi$$

In this system, the Lorentz transformation is

$$(0.6) \quad \Lambda' = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We can write this out explicitly for each coordinate. The time coordinate is the same both \mathcal{S} and \mathcal{S}' since the two frames are at rest relative to each other, so

$$(0.7) \quad \bar{t}' = \gamma(t - \beta x')$$

$$(0.8) \quad \bar{t} = \gamma t - \gamma\beta x \cos \phi - \gamma\beta y \sin \phi$$

The x' coordinate transforms as follows:

$$(0.9) \quad \bar{x}' = \gamma(x' - \beta t)$$

$$(0.10) \quad \bar{x} \cos \phi + \bar{y} \sin \phi = \gamma x \cos \phi + \gamma y \sin \phi - \gamma\beta t$$

And y' transforms as

$$(0.11) \quad \bar{y}' = y'$$

$$(0.12) \quad -\bar{x} \sin \phi + \bar{y} \cos \phi = -x \sin \phi + y \cos \phi$$

Multiplying 0.10 by $\cos \phi$ and 0.12 by $-\sin \phi$ and adding, we get

$$(0.13) \quad \bar{x} = -\beta\gamma t \cos \phi + (\gamma \cos^2 \phi + \sin^2 \phi)x + (\gamma - 1)y \sin \phi \cos \phi$$

Multiplying 0.10 by $\sin \phi$ and 0.12 by $\cos \phi$ and adding, we get

$$(0.14) \quad \bar{y} = -\beta\gamma t \sin \phi + (\gamma - 1)x \sin \phi \cos \phi + (\gamma \sin^2 \phi + \cos^2 \phi)y$$

Combining 0.8, 0.13 and 0.14 (along with $\bar{z} = z$) into a matrix, we get

$$(0.15) \quad \Lambda = \begin{bmatrix} \gamma & -\gamma\beta \cos \phi & -\gamma\beta \sin \phi & 0 \\ -\beta\gamma \cos \phi & \gamma \cos^2 \phi + \sin^2 \phi & (\gamma - 1) \sin \phi \cos \phi & 0 \\ -\beta\gamma \sin \phi & (\gamma - 1) \sin \phi \cos \phi & \gamma \sin^2 \phi + \cos^2 \phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This reduces to 0.6 when $\phi = 0$, so that the axes of the two frames are parallel. This 2-d transformation matrix is also symmetric.