# ELASTIC COLLISION OF TWO IDENTICAL PARTICLES

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Reference: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.59.

Here's another example the kinematics of particle collisions in relativity. This time, we'll look at the common first year physics problem of elastic scattering of two identical masses, with one mass $m$ coming in along the $x$ axis with speed $v$ and hitting another mass $m$ at rest. In classical physics, the two masses always fly off at right angles after the collision, but things are more complicated in relativity. As usual, it's easiest to convert to the centre of momentum frame to do the calculations and then convert back at the end.

The speed of the centre of momentum frame is (I'll set $c=1$ and use the standard symbol $\beta=v/c$ to make notation easier; we can reinsert the $c$ at the end):

$$\bar{\beta} = \frac{\sum p_i}{\sum E_i} \tag{1}$$

$$= \frac{\gamma\beta m}{m+\gamma m} \tag{2}$$

$$= \frac{\gamma\beta}{1+\gamma} \tag{3}$$

where the quantities on the RHS refer to the lab frame. Centre of momentum (COM) quantities have a bar over them. Since mass 2 is at rest in the lab, its velocity in the COM frame is

$$\bar{\beta}_2 = -\frac{\gamma\beta}{1+\gamma} \tag{4}$$

and by conservation of momentum, the COM velocity of mass 1 must be

$$\bar{\beta}_1 = \frac{\gamma\beta}{1+\gamma} \tag{5}$$

*Remark.* At this point, we can check the results using the velocity addition formula. For particle 2, its lab velocity is given by

$$\beta_2 = \frac{\bar{\beta}_2 + \bar{\beta}}{1 + \bar{\beta}_2 \bar{\beta}} = 0 \tag{6}$$

For particle 1, we have

$$\begin{aligned}
\beta_1 &= \frac{\bar{\beta}_1 + \bar{\beta}}{1 + \bar{\beta}_1 \bar{\beta}} && (7)\\
&= 2\frac{\gamma\beta}{1+\gamma}\frac{1}{1+(\gamma\beta)^2/(1+\gamma)^2} && (8)\\
&= 2\frac{\gamma\beta}{1+\gamma}\frac{(1+\gamma)^2}{(1+\gamma)^2+(\gamma\beta)^2} && (9)\\
&= \frac{2\gamma\beta(1+\gamma)}{1+2\gamma+\gamma^2+(\gamma\beta)^2} && (10)\\
&= \frac{2\gamma\beta(1+\gamma)}{\gamma^2(1-\beta^2)+2\gamma+\gamma^2+(\gamma\beta)^2} && (11)\\
&= \frac{2\gamma\beta(1+\gamma)}{2\gamma+2\gamma^2} && (12)\\
&= \beta && (13)
\end{aligned}$$

where in the fifth line we used $\gamma^2\left(1-\beta^2\right)=1$. Thus both velocities convert correctly back to the lab frame.

Returning to our main calculation, what we'd like to do is find the scattering angle in the lab frame, given the angle $\bar{\phi}$ between particle 1's initial and final directions in the COM frame. As the total momentum in COM is zero, we know that the two particles must fly off in opposite directions, so we can write out the $x$ and $y$ components for both particles as

$$\begin{aligned}
\bar{\beta}_{1_x} &= \bar{\beta}\cos\bar{\phi} && (14)\\
\bar{\beta}_{1_y} &= \bar{\beta}\sin\bar{\phi} && (15)\\
\bar{\beta}_{2_x} &= -\bar{\beta}\cos\bar{\phi} && (16)\\
\bar{\beta}_{2_y} &= -\bar{\beta}\sin\bar{\phi} && (17)
\end{aligned}$$

Using the velocity addition formulas, we can convert these back to the lab frame, using $\bar{\gamma} \equiv 1/\sqrt{1-\bar{\beta}^2}$.

$$\beta_{1_x} = \frac{\bar{\beta}_{1_x} + \bar{\beta}}{1 + \bar{\beta}_{1_x}\bar{\beta}} \tag{18}$$

$$= \frac{\bar{\beta}\cos\bar{\phi} + \bar{\beta}}{1 + \bar{\beta}^2\cos\bar{\phi}} \tag{19}$$

$$\beta_{1_y} = \frac{\bar{\beta}_{1_y}}{\bar{\gamma}\left(1 + \bar{\beta}_{1_x}\bar{\beta}\right)} \tag{20}$$

$$= \frac{\bar{\beta}\sin\bar{\phi}}{\bar{\gamma}\left(1 + \bar{\beta}^2\cos\bar{\phi}\right)} \tag{21}$$

$$\beta_{2_x} = \frac{-\bar{\beta}\cos\bar{\phi} + \bar{\beta}}{1 - \bar{\beta}^2\cos\bar{\phi}} \tag{22}$$

$$\beta_{2_y} = \frac{-\bar{\beta}\sin\bar{\phi}}{\bar{\gamma}\left(1 - \bar{\beta}^2\cos\bar{\phi}\right)} \tag{23}$$

In the lab frame, the angle $\phi_1$ that the incident particle makes with its incoming direction is given by

$$\tan\phi_1 = \frac{\beta_{1_y}}{\beta_{1_x}} \tag{24}$$

$$= \frac{\sin\bar{\phi}}{\bar{\gamma}\left(1 + \cos\bar{\phi}\right)} \tag{25}$$

Similarly for particle 2

$$\tan\phi_2 = \frac{\beta_{2_y}}{\beta_{2_x}} \tag{26}$$

$$= \frac{-\sin\bar{\phi}}{\bar{\gamma}\left(1 - \cos\bar{\phi}\right)} \tag{27}$$

Since $\phi_1 > 0$ and $\phi_2 < 0$ (that is, in the lab frame, particle 1 scatters upwards from the $x$ axis while particle 2 scatters downwards; obviously we could interchange the roles of the two particles but to conserve $y$ momentum, the two particles must scatter on opposite sides of the $x$ axis), the total angle between the two particles is $\phi = \phi_1 - \phi_2$. Using the formula for the tangent of the difference of two angles, we get

$$\tan\phi = \frac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1\tan\phi_2} \tag{28}$$

$$= \frac{\sin\bar{\phi}}{\bar{\gamma}}\left[\frac{1}{1+\cos\bar{\phi}}+\frac{1}{1-\cos\bar{\phi}}\right]\left[1-\frac{\sin^2\bar{\phi}}{\bar{\gamma}^2\left(1+\cos\bar{\phi}\right)\left(1-\cos\bar{\phi}\right)}\right]^{-1} \tag{29}$$

$$= \frac{\sin\bar{\phi}}{\bar{\gamma}}\left(\frac{2}{1-\cos^2\bar{\phi}}\right)\left[1-\frac{\sin^2\bar{\phi}}{\bar{\gamma}^2\left(1-\cos^2\bar{\phi}\right)}\right]^{-1} \tag{30}$$

$$= \frac{2}{\bar{\gamma}\sin\bar{\phi}\left(1-1/\bar{\gamma}^2\right)} \tag{31}$$

$$= \frac{2}{\bar{\gamma}\bar{\beta}^2\sin\bar{\phi}} \tag{32}$$

$$\phi = \arctan\left(\frac{2}{\bar{\gamma}\bar{\beta}^2\sin\bar{\phi}}\right) \tag{33}$$

[This answer is the same form as that given in Griffiths's question, except that he seems to use lab coordinates for $\gamma$ and $\beta$. However, after checking my solution I can't see anything wrong, so hopefully I've got it right.]

In the classical limit, $\beta$ becomes very small and $\gamma \to 1$ so from 3, $\bar{\beta}$ also becomes very small and $\bar{\gamma} \to 1$, so the argument of the arctan tends to infinity and $\phi \to \pi/2$ as required.