MOTION UNDER A CONSTANT MINKOWSKI FORCE

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Reference: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.60.

The Minkowski force **K** is the rate of change of four-momentum with respect to proper time, and allows Newton's law to be written in its natural form

$$\mathbf{K}=m\alpha \tag{1}$$

where α is the proper acceleration, or second derivative of position with respect to proper time. Here we'll investigate the behaviour of a particle subject to a constant Minkowski force in one dimension.

In terms of ordinary force, we have

$$K=\frac{dp}{d\tau}=\frac{dp}{dt}\frac{dt}{d\tau}=\frac{1}{\sqrt{1-u^{2}/c^{2}}}F \tag{2}$$

The ordinary momentum p is

$$p=\frac{mu}{\sqrt{1-u^{2}/c^{2}}} \tag{3}$$

so its derivative is

$$\frac{dp}{dt}=\frac{m}{\sqrt{1-u^{2}/c^{2}}}\frac{du}{dt}+\frac{mu^{2}}{\left(1-u^{2}/c^{2}\right)^{3/2}}\frac{du}{dt} \tag{4}$$

Inserting this into 2 we get

$$\frac{K}{m}dt=\frac{du}{1-u^{2}/c^{2}}+\frac{u^{2}\,du}{\left(1-u^{2}/c^{2}\right)^{2}} \tag{5}$$

We can integrate both sides (using software, or integral tables) to get

$$\frac{K}{m}t+C=\frac{c}{4}\ln\left[\frac{c+u}{c-u}\right]+\frac{c^{2}}{4}\left[\frac{1}{c-u}-\frac{1}{c+u}\right] \tag{6}$$

where C is a constant of integration. If the initial conditions are $u=0$ at $t=0$, then $C=0$ and we have

$$\frac{K}{m}t=\frac{c}{4}\ln\left[\frac{c+u}{c-u}\right]+\frac{c^2}{4}\left[\frac{1}{c-u}-\frac{1}{c+u}\right] \tag{7}$$

This is an implicit equation for the speed of the particle as a function of time. If we want the position as a function of time, we need a relation between u and x. Returning to 2 and 3 we have

$$\sqrt{1-u^2/c^2}\frac{K}{m}=\frac{d}{dt}\left(\frac{u}{\sqrt{1-u^2/c^2}}\right) \tag{8}$$

We can use the chain rule to convert the derivative on the RHS to a derivative with respect to x by multiplying both sides by dt/dx

$$\frac{dt}{dx}\sqrt{1-u^2/c^2}\frac{K}{m}=\frac{dt}{dx}\frac{d}{dt}\left(\frac{u}{\sqrt{1-u^2/c^2}}\right)=\frac{d}{dx}\left(\frac{u}{\sqrt{1-u^2/c^2}}\right) \tag{9}$$

Now $dx/dt=u$ so $dt/dx=1/u$ and

$$\frac{\sqrt{1-u^2/c^2}}{u}\frac{K}{m}=\frac{d}{dx}\left(\frac{u}{\sqrt{1-u^2/c^2}}\right) \tag{10}$$

If we call the expression in the parentheses on the RHS A, then we can integrate with respect to x (since K/m is a constant):

$$\begin{aligned} A &\equiv \frac{u}{\sqrt{1-u^2/c^2}} && (11)\\ \frac{1}{A}\frac{K}{m} &= \frac{dA}{dx} && (12)\\ \frac{K}{m}x+C &= \frac{1}{2}A^2 && (13)\end{aligned}$$

Again, starting from rest at the origin we have $u=0$ when $x=0$ so $A=0$ also, and therefore $C=0$, so we have

$$A=\frac{u}{\sqrt{1-u^2/c^2}}=\sqrt{\frac{2Kx}{m}} \tag{14}$$

At this point we could get a relation between x and t by solving 14 for u in terms of x and then substituting this into 7. For reference, we get

$$u=\sqrt{\frac{2Kx}{m}}\frac{1}{\sqrt{1+2Kx/mc^{2}}} \tag{15}$$

so substituting will give something of a mess. To get the answer given in Griffiths requires a bit of algebra, but here is how I did it. Griffiths defines the quantity z as

$$\begin{aligned} z &\equiv \sqrt{\frac{2Kx}{mc^{2}}} && (16)\\ &= \frac{A}{c} && (17)\\ &= \frac{u}{c\sqrt{1-u^{2}/c^{2}}} && (18)\end{aligned}$$

The quantities appearing in Griffiths's answer are

$$\begin{aligned} \sqrt{1+z^{2}} &= \frac{c}{\sqrt{c^{2}-u^{2}}} && (19)\\ z\sqrt{1+z^{2}} &= \frac{u}{c\left(1-u^{2}/c^{2}\right)} && (20)\end{aligned}$$

We can rewrite 7 to get

$$\frac{2Kt}{mc}=\frac{1}{2}\ln\left[\frac{c+u}{c-u}\right]+\frac{c}{2}\left[\frac{1}{c-u}-\frac{1}{c+u}\right] \tag{21}$$

We'll deal with the logarithm first. Its argument is

$$\begin{aligned} \frac{c+u}{c-u} &= \frac{(c+u)^{2}}{c^{2}\left(1-u^{2}/c^{2}\right)} && (22)\\ &= \frac{2u}{c\left(1-u^{2}/c^{2}\right)}+\frac{u^{2}+c^{2}}{c^{2}-u^{2}} && (23)\\ &= \frac{2u}{c\left(1-u^{2}/c^{2}\right)}+\frac{c^{2}-u^{2}+2u^{2}}{c^{2}-u^{2}} && (24)\\ &= \frac{2u}{c\left(1-u^{2}/c^{2}\right)}+1+\frac{2u^{2}}{c^{2}\left(1-u^{2}/c^{2}\right)} && (25)\end{aligned}$$

Now we also have

$$\left(z+\sqrt{1+z^2}\right)^2 = 2z^2+2z\sqrt{1+z^2}+1 \tag{26}$$

$$= \frac{2u^2}{c^2\left(1-u^2/c^2\right)}+\frac{2u}{c\left(1-u^2/c^2\right)}+1 \tag{27}$$

$$= \frac{c+u}{c-u} \tag{28}$$

Therefore

$$\frac{1}{2}\ln\left[\frac{c+u}{c-u}\right] = \ln\sqrt{\frac{c+u}{c-u}} \tag{29}$$

$$= \ln\left(z+\sqrt{1+z^2}\right) \tag{30}$$

For the second term in 21, we have

$$\frac{c}{2}\left[\frac{1}{c-u}-\frac{1}{c+u}\right] = \frac{c}{2}\frac{2u}{c^2\left(1-u^2/c^2\right)} \tag{31}$$

$$= \frac{u}{c\left(1-u^2/c^2\right)} \tag{32}$$

$$= z\sqrt{1+z^2} \tag{33}$$

Putting it all together, we have

$$\frac{2Kt}{mc} = \ln\left(z+\sqrt{1+z^2}\right)+z\sqrt{1+z^2} \tag{34}$$