

## MOTION UNDER A CONSTANT MINKOWSKI FORCE

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Reference: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.60.

The Minkowski force  $\mathbf{K}$  is the rate of change of four-momentum with respect to proper time, and allows Newton's law to be written in its natural form

$$(0.1) \quad \mathbf{K} = m\alpha$$

where  $\alpha$  is the proper acceleration, or second derivative of position with respect to proper time. Here we'll investigate the behaviour of a particle subject to a constant Minkowski force in one dimension.

In terms of ordinary force, we have

$$(0.2) \quad K = \frac{dp}{d\tau} = \frac{dp}{dt} \frac{dt}{d\tau} = \frac{1}{\sqrt{1-u^2/c^2}} F$$

The ordinary momentum  $p$  is

$$(0.3) \quad p = \frac{mu}{\sqrt{1-u^2/c^2}}$$

so its derivative is

$$(0.4) \quad \frac{dp}{dt} = \frac{m}{\sqrt{1-u^2/c^2}} \frac{du}{dt} + \frac{mu^2}{(1-u^2/c^2)^{3/2}} \frac{du}{dt}$$

Inserting this into 0.2 we get

$$(0.5) \quad \frac{K}{m} dt = \frac{du}{1-u^2/c^2} + \frac{u^2 du}{(1-u^2/c^2)^2}$$

We can integrate both sides (using software, or integral tables) to get

$$(0.6) \quad \frac{K}{m} t + C = \frac{c}{4} \ln \left[ \frac{c+u}{c-u} \right] + \frac{c^2}{4} \left[ \frac{1}{c-u} - \frac{1}{c+u} \right]$$

where  $C$  is a constant of integration. If the initial conditions are  $u = 0$  at  $t = 0$ , then  $C = 0$  and we have

$$(0.7) \quad \frac{K}{m}t = \frac{c}{4} \ln \left[ \frac{c+u}{c-u} \right] + \frac{c^2}{4} \left[ \frac{1}{c-u} - \frac{1}{c+u} \right]$$

This is an implicit equation for the speed of the particle as a function of time. If we want the position as a function of time, we need a relation between  $u$  and  $x$ . Returning to 0.2 and 0.3 we have

$$(0.8) \quad \sqrt{1-u^2/c^2} \frac{K}{m} = \frac{d}{dt} \left( \frac{u}{\sqrt{1-u^2/c^2}} \right)$$

We can use the chain rule to convert the derivative on the RHS to a derivative with respect to  $x$  by multiplying both sides by  $dt/dx$

$$(0.9) \quad \frac{dt}{dx} \sqrt{1-u^2/c^2} \frac{K}{m} = \frac{dt}{dx} \frac{d}{dt} \left( \frac{u}{\sqrt{1-u^2/c^2}} \right) = \frac{d}{dx} \left( \frac{u}{\sqrt{1-u^2/c^2}} \right)$$

Now  $dx/dt = u$  so  $dt/dx = 1/u$  and

$$(0.10) \quad \frac{\sqrt{1-u^2/c^2} K}{u m} = \frac{d}{dx} \left( \frac{u}{\sqrt{1-u^2/c^2}} \right)$$

If we call the expression in the parentheses on the RHS  $A$ , then we can integrate with respect to  $x$  (since  $K/m$  is a constant):

$$(0.11) \quad A \equiv \frac{u}{\sqrt{1-u^2/c^2}}$$

$$(0.12) \quad \frac{1}{A} \frac{K}{m} = \frac{dA}{dx}$$

$$(0.13) \quad \frac{K}{m}x + C = \frac{1}{2}A^2$$

Again, starting from rest at the origin we have  $u = 0$  when  $x = 0$  so  $A = 0$  also, and therefore  $C = 0$ , so we have

$$(0.14) \quad A = \frac{u}{\sqrt{1-u^2/c^2}} = \sqrt{\frac{2Kx}{m}}$$

At this point we could get a relation between  $x$  and  $t$  by solving 0.14 for  $u$  in terms of  $x$  and then substituting this into 0.7. For reference, we get

$$(0.15) \quad u = \sqrt{\frac{2Kx}{m}} \frac{1}{\sqrt{1 + 2Kx/mc^2}}$$

so substituting will give something of a mess. To get the answer given in Griffiths requires a bit of algebra, but here is how I did it. Griffiths defines the quantity  $z$  as

$$(0.16) \quad z \equiv \sqrt{\frac{2Kx}{mc^2}}$$

$$(0.17) \quad = \frac{A}{c}$$

$$(0.18) \quad = \frac{u}{c\sqrt{1 - u^2/c^2}}$$

The quantities appearing in Griffiths's answer are

$$(0.19) \quad \sqrt{1 + z^2} = \frac{c}{\sqrt{c^2 - u^2}}$$

$$(0.20) \quad z\sqrt{1 + z^2} = \frac{u}{c(1 - u^2/c^2)}$$

We can rewrite 0.7 to get

$$(0.21) \quad \frac{2Kt}{mc} = \frac{1}{2} \ln \left[ \frac{c+u}{c-u} \right] + \frac{c}{2} \left[ \frac{1}{c-u} - \frac{1}{c+u} \right]$$

We'll deal with the logarithm first. Its argument is

$$(0.22) \quad \frac{c+u}{c-u} = \frac{(c+u)^2}{c^2(1 - u^2/c^2)}$$

$$(0.23) \quad = \frac{2u}{c(1 - u^2/c^2)} + \frac{u^2 + c^2}{c^2 - u^2}$$

$$(0.24) \quad = \frac{2u}{c(1 - u^2/c^2)} + \frac{c^2 - u^2 + 2u^2}{c^2 - u^2}$$

$$(0.25) \quad = \frac{2u}{c(1 - u^2/c^2)} + 1 + \frac{2u^2}{c^2(1 - u^2/c^2)}$$

Now we also have

$$(0.26) \quad \left(z + \sqrt{1+z^2}\right)^2 = 2z^2 + 2z\sqrt{1+z^2} + 1$$

$$(0.27) \quad = \frac{2u^2}{c^2(1-u^2/c^2)} + \frac{2u}{c(1-u^2/c^2)} + 1$$

$$(0.28) \quad = \frac{c+u}{c-u}$$

Therefore

$$(0.29) \quad \frac{1}{2} \ln \left[ \frac{c+u}{c-u} \right] = \ln \sqrt{\frac{c+u}{c-u}}$$

$$(0.30) \quad = \ln \left( z + \sqrt{1+z^2} \right)$$

For the second term in 0.21, we have

$$(0.31) \quad \frac{c}{2} \left[ \frac{1}{c-u} - \frac{1}{c+u} \right] = \frac{c}{2} \frac{2u}{c^2(1-u^2/c^2)}$$

$$(0.32) \quad = \frac{u}{c(1-u^2/c^2)}$$

$$(0.33) \quad = z\sqrt{1+z^2}$$

Putting it all together, we have

$$(0.34) \quad \frac{2Kt}{mc} = \ln \left( z + \sqrt{1+z^2} \right) + z\sqrt{1+z^2}$$