SELF-FORCE ON A DIPOLE IN HYPERBOLIC MOTION

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Reference: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.61.

Here we'll revisit the problem of calculating the self-force of a moving charge. In our earlier derivation, we considered a single charge q split into two equal charges $q/2$ separated by a distance d and moving perpendicular to the line joining the two charges. If the motion is in the x direction and the line joining the charges is parallel to the z axis, the only net electric field felt by one charge due to the field generated by the other charge at a retarded time t_r is

$$E_x = \frac{q}{8\pi\varepsilon_0 c^3}\frac{r}{\left(r-\frac{lv}{c}\right)^3}\left[\left(\frac{cl}{r}-v\right)\left(\frac{c^2}{\gamma^2}+la\right)-ac\left(r-\frac{lv}{c}\right)\right] \tag{1}$$

where r is the distance from the other charge at the retarded time to the current charge at the current time, l is the distance moved in the x direction in the time $t-t_r$ and v and a are the velocity and acceleration at the retarded time. $\gamma = 1/\sqrt{1-v^2/c^2}$ as usual. The calculating proceeded from here by assuming that the separation distance d was small and deriving the self-force in that limiting case.

We'll now consider an electric dipole consisting of charges $+q$ and $-q$ separated by a distance d, but without assuming d to be small. The electric field due to $+q$ at the retarded time t_r felt by $-q$ at the current time t is therefore

$$\begin{aligned} E_x &= \frac{q}{4\pi\varepsilon_0 c^3}\frac{r}{\left(r-\frac{lv}{c}\right)^3}\left[\left(\frac{cl}{r}-v\right)\left(\frac{c^2}{\gamma^2}+la\right)-ac\left(r-\frac{lv}{c}\right)\right] && (2)\\ &= \frac{q}{4\pi\varepsilon_0 c^2}\frac{1}{\left(r-\frac{lv}{c}\right)^3}\left[\frac{c^2 l}{\gamma^2}+al^2-\frac{cvr}{\gamma^2}-ar^2\right] && (3) \end{aligned}$$

where we've changed the 8 in the denominator to a 4 because we've replaced $q/2$ by q. To go any further, we need to make some assumptions about the motion of the dipole, so we'll suppose that it is moving under hyperbolic motion. Griffiths wants us to consider the position of the charge to be given by

$$x(t) = \frac{mc^2}{F}\sqrt{1+(Ft/mc)^2}-1 \tag{4}$$

However, to simplify the notation a bit, we'll consider the more general form

$$x(t) = \sqrt{b^2+(ct)^2} \tag{5}$$

where b is a constant with dimensions of length. This is equivalent to 4 except that $x = b$ at $t = 0$ rather than $x = 0$. Since all calculations depend only on how much the dipole moves over a time interval and not on its absolute position, this change won't affect anything that follows.

Using this form, we can calculate the velocity and acceleration by taking derivatives:

$$\begin{aligned}
v(t) &= \frac{c^2 t}{\sqrt{b^2+(ct)^2}} = \frac{c^2 t}{x} && (6)\\
a(t) &= \frac{c^2}{\sqrt{b^2+(ct)^2}} - \frac{c^4 t^2}{\left(b^2+(ct)^2\right)^{3/2}} && (7)\\
&= \frac{b^2 c^2}{\left(b^2+(ct)^2\right)^{3/2}} = \frac{b^2 c^2}{x^3} && (8)\\
\frac{1}{\gamma^2} &= 1-\frac{v^2}{c^2} && (9)\\
&= \frac{b^2}{x^2} && (10)
\end{aligned}$$

Adding a subscript r to indicate a quantity evaluated at t_r and plugging these equations into 3, we get

$$\frac{4\pi\varepsilon_0 c^2}{q}E_x = \left(r-\frac{lct_r}{x_r}\right)^{-3}\frac{c^2 b^2}{x_r^2}\left[l+\frac{l^2}{x_r}-\frac{crt_r}{x_r}-\frac{r^2}{x_r}\right] \tag{11}$$

We now need to express r, l and x_r in terms of t_r. From the geometry of the setup

$$r^2 = l^2 + d^2 \tag{12}$$

and since a signal travels from $+q$ at time t_r to $-q$ at time t over a distance r, we have

$$r = c(t-t_r) \tag{13}$$
$$l = \sqrt{c^2(t-t_r)^2-d^2} \tag{14}$$

Substituting these into 11 and simplifying gives

$$\frac{4\pi\varepsilon_0}{qb^2}E_x = \frac{c^2t_r(t-t_r)+d^2-x_r\sqrt{c^2(t-t_r)^2-d^2}}{c^3\left(t_r\sqrt{c^2(t-t_r)^2-d^2}-x_rt+x_rt_r\right)^3} \tag{15}$$
$$= \frac{c^2t_r(t-t_r)+d^2-\sqrt{b^2+(ct_r)^2}\sqrt{c^2(t-t_r)^2-d^2}}{c^3\left(t_r\sqrt{c^2(t-t_r)^2-d^2}-\sqrt{b^2+(ct)^2}(t-t_r)\right)^3} \tag{16}$$

where we used 5 to get rid of x_r in the last line.

Also, l is the distance moved in time $t-t_r$, so

$$l = x(t)-x(t_r) \tag{17}$$
$$= \sqrt{b^2+(ct)^2}-\sqrt{b^2+(ct_r)^2} \tag{18}$$

[Note that we can't take $l = v(t-t_r)$, since the velocity is changing over that time interval.] We can therefore find t in terms of t_r by solving

$$\sqrt{c^2(t-t_r)^2-d^2} = \sqrt{b^2+(ct)^2}-\sqrt{b^2+(ct_r)^2} \tag{19}$$

This turns out to be a quadratic equation, with solutions

$$t = t_r+\frac{1}{2cb^2t_r}\left[ct_r^2d^2 \pm dt_r\sqrt{(4b^2+d^2)(c^2t_r^2+b^2)}\right] \tag{20}$$

To decide which sign to take, we note that for very small d, the square root term dominates since the first term is of order d^2. Since we must have $t > t_r$, we'll need to take the + sign. We then get for l:

$$l = \sqrt{c^2\left(t-t_r\right)^2 - d^2} \tag{21}$$

$$= \frac{d}{2b^2}\sqrt{\left(4c^2b^2t_r^2 + 2c^2d^2t_r^2 + d^2b^2 + 2cdt_r\sqrt{\left(4b^2+d^2\right)\left(c^2t_r^2+b^2\right)}\right)} \tag{22}$$

$$= \frac{d}{2b^2}\sqrt{\left(ct_r\sqrt{\left(4b^2+d^2\right)} + d\sqrt{c^2t_r^2+b^2}\right)^2} \tag{23}$$

$$= \frac{d}{2b^2}\left(ct_r\sqrt{\left(4b^2+d^2\right)} + d\sqrt{c^2t_r^2+b^2}\right) \tag{24}$$

Plugging this into 16 we get

$$\frac{4\pi\varepsilon_0}{qb^2}E_x = \frac{c^2t_r\left(t-t_r\right) + d^2 - \frac{d}{2b^2}\left(ct_r\sqrt{\left(4b^2+d^2\right)} + d\sqrt{c^2t_r^2+b^2}\right)\sqrt{b^2+\left(ct_r\right)^2}}{c^3\left(t_r\frac{d}{2b^2}\left(ct_r\sqrt{\left(4b^2+d^2\right)} + d\sqrt{c^2t_r^2+b^2}\right) - \sqrt{b^2+\left(ct\right)^2}\left(t-t_r\right)\right)^3} \tag{25}$$

$$= \frac{4b^4}{c^3}\frac{t_r^2c^2d^2 - 2c^2b^2t_r\left(t-t_r\right) - d^2b^2 + ct_rd\sqrt{\left(4b^2+d^2\right)}\sqrt{c^2t_r^2+b^2}}{\left[\sqrt{c^2t_r^2+b^2}\left(2b^2\left(t-t_r\right) - d^2t_r\right) - cdt_r^2\sqrt{\left(4b^2+d^2\right)}\right]^3} \tag{26}$$

We can now substitute from 20

$$t - t_r = \frac{1}{2cb^2t_r}\left[ct_r^2d^2 + dt_r\sqrt{\left(4b^2+d^2\right)\left(c^2t_r^2+b^2\right)}\right] \tag{27}$$

The numerator in 26 is now

$$t_r^2c^2d^2 - c\left[ct_r^2d^2 + dt_r\sqrt{\left(4b^2+d^2\right)\left(c^2t_r^2+b^2\right)}\right] - d^2b^2 + ct_rd\sqrt{\left(4b^2+d^2\right)}\sqrt{c^2t_r^2+b^2} = -d^2b^2 \tag{28}$$

and the denominator is

$$\left[\sqrt{c^2t_r^2+b^2}\left(\frac{1}{ct_r}\left[ct_r^2d^2+dt_r\sqrt{(4b^2+d^2)\left(c^2t_r^2+b^2\right)}\right]-d^2t_r\right)-cdt_r^2\sqrt{(4b^2+d^2)}\right]^3 = \tag{29}$$

$$\left[\sqrt{(4b^2+d^2)}\left(\frac{d}{c}\left(c^2t_r^2+b^2\right)-cdt_r^2\right)\right]^3 = \tag{30}$$

$$\frac{b^6d^3}{c^3}\left(4b^2+d^2\right)^{3/2} \tag{31}$$

Putting this back into 26 we get

$$\begin{aligned}
\frac{4\pi\varepsilon_0}{qb^2}E_x &= \frac{4b^4}{c^3}\left(-\frac{d^2b^2}{\frac{b^6d^3}{c^3}\left(4b^2+d^2\right)^{3/2}}\right) && (32)\\
&= -\frac{4}{d\left(4b^2+d^2\right)^{3/2}} && (33)\\
E_x &= -\frac{qb^2}{\pi\varepsilon_0 d\left(4b^2+d^2\right)^{3/2}} && (34)\\
&= -\frac{q}{8\pi\varepsilon_0 db\left(1+(d/2b)^2\right)^{3/2}} && (35)
\end{aligned}$$

Miraculously, the dependence on both t and t_r has disappeared, showing that the field (and therefore the self-force) is constant. To get this answer back into the form given by Griffiths, we compare 4 and 5.

$$\begin{aligned}
x &= \sqrt{b^2+c^2t^2} && (36)\\
&= b\sqrt{1+\left(\frac{ct}{b}\right)^2} && (37)\\
&= \frac{mc^2}{F}\sqrt{1+(Ft/mc)^2} && (38)\\
b &= \frac{mc^2}{F} && (39)
\end{aligned}$$

[We can ignore the -1 in 4 since that just changes the origin of x, and all the calculations above depend only on $x_r - x$ so the -1 cancels out.] With this value for b we have

$$E_x = -\frac{qF}{8\pi\varepsilon_0 mc^2 d\left(1+(Fd/2mc^2)^2\right)^{3/2}} \tag{40}$$

The force felt by $-q$ is therefore

$$F_x = -qE_x = \frac{q^2 F}{8\pi\varepsilon_0 mc^2 d\left(1+(Fd/2mc^2)^2\right)^{3/2}} \tag{41}$$

The total self-force on the dipole is twice this, since there is an equal force on $+q$ due to $-q$, so

$$F_{tot} = \frac{q^2 F}{4\pi\varepsilon_0 mc^2 d\left(1+(Fd/2mc^2)^2\right)^{3/2}} \tag{42}$$

[However, it would seem to me that there should also be forces on each charge due to their own fields at the retarded time. If these forces are equal in magnitude to the cross-self-force calculated here, they should also be opposite in direction since the force is between the same charge at different times, causing repulsion. Thus it would seem that the total force is zero, which actually makes more sense than having the dipole constantly accelerating without any external force. What am I missing?]

In any case, if we set $F_{tot} = F$ and solve for F, we get

$$F = \frac{2mc^2}{d}\sqrt{\left(\frac{q^2}{4\pi\varepsilon_0 mc^2 d}\right)^{2/3} - 1} \tag{43}$$

$$= \frac{2mc^2}{d}\sqrt{\left(\frac{\mu_0 q^2}{4\pi md}\right)^{2/3} - 1} \tag{44}$$