MAXWELL'S EQUATIONS IN MATTER: BOUNDARY CONDITIONS

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Section 7.3.6.

Maxwell's equations in matter are

$$\nabla\cdot\mathbf{D} = \rho_f \tag{1}$$
$$\nabla\cdot\mathbf{B} = 0 \tag{2}$$
$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{3}$$
$$\nabla\times\mathbf{H} = \mathbf{J}_f+\frac{\partial\mathbf{D}}{\partial t} \tag{4}$$

We can write these in integral form using the divergence theorem and Stokes's theorem:

$$\int_{\mathcal{S}}\mathbf{D}\cdot d\mathbf{a} = Q_f \tag{5}$$
$$\int_{\mathcal{S}}\mathbf{B}\cdot d\mathbf{a} = 0 \tag{6}$$
$$\int_{\mathcal{P}}\mathbf{E}\cdot d\ell = -\frac{d}{dt}\int_{\mathcal{A}}\mathbf{B}\cdot d\mathbf{a} \tag{7}$$
$$\int_{\mathcal{P}}\mathbf{H}\cdot d\ell = \int_{\mathcal{A}}\mathbf{J}_f\cdot d\mathbf{a}+\int_{\mathcal{A}}\mathbf{D}\cdot d\mathbf{a} \tag{8}$$
$$= I_f+\frac{d}{dt}\int_{\mathcal{A}}\mathbf{D}\cdot d\mathbf{a} \tag{9}$$

Here, Q_f is the free charge enclosed by closed surface $\mathcal{S}$, I_f is the free current passing through closed loop $\mathcal{P}$ and $\mathcal{A}$ is any surface bounded by $\mathcal{P}$.

To work out the boundary conditions at the interface between two media, where medium i has permittivity ε_i and permeability μ_i we can look at the first equation for $\mathbf{D}$. Choosing a Gaussian pillbox of infinitesimal thickness straddling the the boundary, and taking the area of the top and bottom of the pillbox as $d\mathbf{a}$, we have

$$\mathbf{D}_1 \cdot d\mathbf{a} - \mathbf{D}_2 \cdot d\mathbf{a} = \sigma_f da \tag{10}$$

where σ_f is the free charge density at the boundary. Any volume free charge doesn't contribute since we can make the pillbox as thin as we like (for the same reason, the sides of the pillbox don't contribute since they become vanishly small). The minus sign occurs because $d\mathbf{a}$ points up on side 1 and down on side 2.

The product $\mathbf{D}_1 \cdot d\mathbf{a} = D_1^{\perp} da$ where $D_1^{\perp}$ is the component of $\mathbf{D}_1$ perpendicular to the surface, so we get

$$D_1^{\perp} - D_2^{\perp} = \sigma_f \tag{11}$$

This is the same condition as the one for electrostatics and shows that the perpendicular component of $\mathbf{D}$ is discontinuous at a boundary if there is free charge at the boundary. The same calculation for $\mathbf{B}$ gives

$$B_1^{\perp} - B_2^{\perp} = 0 \tag{12}$$

so the perpendicular component of $\mathbf{B}$ is continuous in all cases (unless magnetic monopoles exist!).

For the path integrals, we can choose a rectangular loop with a side length of ℓ parallel to the surface and an infinitesimal length perpendicular to the surface. If we orient the loop so its face is perpendicular to the boundary and do the line integral around it, we get for $\mathbf{E}$:

$$\mathbf{E}_1 \cdot \ell - \mathbf{E}_2 \cdot \ell = -\frac{d}{dt} \int_{\mathcal{A}} \mathbf{B} \cdot d\mathbf{a} \tag{13}$$

In the limit as the area of the loop goes to zero, the integral on the right goes to zero since the area $\mathcal{A}$ over which the integral is done goes to zero. If we choose the boundary to lie in the xy plane, we can do this calculation for ℓ parallel first to the x axis and then to the y axis, so we get

$$E_{1x} - E_{2x} = 0 \tag{14}$$

$$E_{1y} - E_{2y} = 0 \tag{15}$$

or with $\mathbf{E}_i^{\|} \equiv E_{ix}\hat{\mathbf{x}} + E_{iy}\hat{\mathbf{y}}$ being the component of $\mathbf{E}_i$ parallel to the surface:

$$\mathbf{E}_1^{\|} - \mathbf{E}_2^{\|} = 0 \tag{16}$$

For the $\mathbf{H}$ integral, we can imagine the free current I_f as made up from a surface current $\mathbf{K}_f$ that flows along the boundary. This current may not be perpendicular to the face of the loop, but since the face of the loop is normal to the boundary surface, if we have a unit vector normal to the loop's face we can take the dot product of $\mathbf{K}_f$ with this vector to find the component of the surface current density that flows through the loop. Griffiths does this by taking $\hat{\mathbf{n}}$ to be a unit normal to the boundary surface, so that $\hat{\mathbf{n}}\times\ell$ is a vector with magnitude ℓ normal to the face of the loop. The total current flowing through the loop is then

$$I_f = \mathbf{K}_f\cdot(\hat{\mathbf{n}}\times\ell) = \left(\mathbf{K}_f\times\hat{\mathbf{n}}\right)\cdot\ell \tag{17}$$

since we can interchange the dot and cross in a vector triple product.

In the limit of an infinitesimally thin loop, the surface integral $\int_{\mathcal{A}}\mathbf{D}\cdot d\mathbf{a}$ goes to zero, so we're left with

$$\mathbf{H}_1\cdot\ell-\mathbf{H}_2\cdot\ell = \left(\mathbf{K}_f\times\hat{\mathbf{n}}\right)\cdot\ell \tag{18}$$

Since this must be true for arbitrary ℓ we again get a condition on $\mathbf{H}_i^{\|}$:

$$\mathbf{H}_1^{\|}-\mathbf{H}_2^{\|} = \mathbf{K}_f\times\hat{\mathbf{n}} \tag{19}$$

If the media are both linear and homogeneous, then

$$\mathbf{D}_i = \varepsilon_i\mathbf{E}_i \tag{20}$$

$$\mathbf{H}_i = \frac{1}{\mu_i}\mathbf{B}_i \tag{21}$$

so

$$\varepsilon_1 E_1^{\perp}-\varepsilon_2 E_2^{\perp} = \sigma_f \tag{22}$$

$$B_1^{\perp}-B_2^{\perp} = 0 \tag{23}$$

$$\mathbf{E}_1^{\|}-\mathbf{E}_2^{\|} = 0 \tag{24}$$

$$\frac{1}{\mu_1}\mathbf{B}_1^{\|}-\frac{1}{\mu_2}\mathbf{B}_2^{\|} = \mathbf{K}_f\times\hat{\mathbf{n}} \tag{25}$$

Finally, in cases where there is no surface charge or current at the boundary:

$$
\begin{aligned}
\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} &= 0 &(26)\\
B_1^{\perp} - B_2^{\perp} &= 0 &(27)\\
\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} &= 0 &(28)\\
\frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} &= 0 &(29)
\end{aligned}
$$

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