ANGULAR MOMENTUM - EIGENFUNCTIONS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Post date: 1 Sep 2011.

Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Sec. 4.3.2 & Problem 4.21.

We have seen how to calculate the eigenvalues of the angular momentum operators L^2 and L_z using only the commutation relations for the three components of $\mathbf{L}$. This doesn't tell us the eigenfunctions, however, so we'll consider that here. The procedure unfortunately involves a lot of calculation, so we just have to get on with it.

It is easiest to work in spherical coordinates, so we first need to express $\mathbf{L}$ in spherical coordinates. We have

$$\begin{aligned}
\mathbf{L} &= -i\hbar\mathbf{r}\times\nabla \\
\nabla &= \hat{r}\frac{\partial}{\partial r}+\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}+\hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}
\end{aligned}$$

The radius vector is $\mathbf{r}=r\hat{r}$ and the cross product of any vector with itself is zero, so the angular momentum operator becomes

$$\begin{aligned}
\mathbf{L} &= -i\hbar\left[\hat{r}\times\hat{\theta}\frac{\partial}{\partial\theta}+\hat{r}\times\hat{\phi}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right] \\
&= -i\hbar\left[\hat{\phi}\frac{\partial}{\partial\theta}-\hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right]
\end{aligned}$$

using the cross product relations for the spherical unit vectors $\hat{r}\times\hat{\theta}=\hat{\phi}$ and $\hat{r}\times\hat{\phi}=-\hat{\theta}$.

We need to extract from this formula the values for the three components of $\mathbf{L}$, so we need to convert this equation so that the vectors are the cartesian unit vectors. To do this, we can use the formulas

$$\begin{aligned}
\hat{\theta} &= \cos\theta\cos\phi\hat{i}+\cos\theta\sin\phi\hat{j}-\sin\theta\hat{k} \\
\hat{\phi} &= -\sin\phi\hat{i}+\cos\phi\hat{j}
\end{aligned}$$

Plugging these into the equation for $\mathbf{L}$ and collecting terms, we get

$$\begin{aligned}
\mathbf{L} &= -i\hbar\left[(-\sin\phi\hat{i}+\cos\phi\hat{j})\frac{\partial}{\partial\theta}-(\cos\theta\cos\phi\hat{i}+\cos\theta\sin\phi\hat{j}-\sin\theta\hat{k})\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right]\\
L_x &= -i\hbar\left[-\sin\phi\frac{\partial}{\partial\theta}-\cot\theta\cos\phi\frac{\partial}{\partial\phi}\right]\\
L_y &= -i\hbar\left[\cos\phi\frac{\partial}{\partial\theta}-\cot\theta\sin\phi\frac{\partial}{\partial\phi}\right]\\
L_z &= -i\hbar\frac{\partial}{\partial\phi}
\end{aligned}$$

The raising and lowering operators can be worked out from these formulas, and we get

$$\begin{aligned}
L_\pm &= L_x \pm iL_y\\
&= -i\hbar\left[(-\sin\phi\pm i\cos\phi)\frac{\partial}{\partial\theta}-\cot\theta(\cos\phi\pm i\sin\phi)\frac{\partial}{\partial\phi}\right]\\
&= \hbar\left[\pm(\cos\phi\pm i\sin\theta)\frac{\partial}{\partial\theta}+i\cot\theta(\cos\phi\pm i\sin\phi)\frac{\partial}{\partial\phi}\right]\\
&= \hbar\left[\pm e^{\pm i\phi}\frac{\partial}{\partial\theta}+i\cot\phi e^{\pm\phi}\frac{\partial}{\partial\phi}\right]\\
&= \pm\hbar e^{\pm i\phi}\left[\frac{\partial}{\partial\theta}\pm i\cot\phi\frac{\partial}{\partial\phi}\right]
\end{aligned}$$

We now have all the bits we need to calculate L^2. We use the formula

$$L^2 = L_+L_- + L_z^2 - \hbar L_z$$

We work out the first term by applying the two operators to a test function f:

$$\begin{aligned}
L_+L_-f &= -\hbar^2 e^{i\phi}\left(\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right)\left(e^{-i\phi}\left(\frac{\partial f}{\partial\theta}-i\cot\theta\frac{\partial f}{\partial\phi}\right)\right)\\
&= -\hbar^2(f_{\theta\theta}+i\csc^2\theta f_\phi - i\cot\theta f_{\phi\theta}+\\
&\quad i\cot\theta(-i(f_\theta - i\cot\theta f_\phi)+f_{\phi\theta}-i\cot\theta f_{\phi\phi}))\\
&= -\hbar^2(f_{\theta\theta}-\cot\theta f_\theta+\cot^2\theta f_{\phi\phi}+i(\csc^2\theta-\cot^2\theta)f_\phi)
\end{aligned}$$

Subscripts on the test function f represent derivatives with respect to the subscripted variable. Using the trig identity $\csc^2\theta-\cot^2\theta=1$, we get the required result after removing the test function:

$$L_+L_- = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right)$$

We now work out L^2:

$$\begin{aligned} L^2 &= L_+L_- + L_z^2 - \hbar L_z \\ &= -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right) - \hbar^2\frac{\partial^2}{\partial\phi^2} + \hbar^2 i\frac{\partial}{\partial\phi} \\ &= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \end{aligned}$$

using the trig identity $1+\cot^2\theta = 1/\sin^2\theta$ to simplify the second derivative term in ϕ.

The eigenfunction $f_l^m(\theta,\phi)$ must satisfy

$$\begin{aligned} L^2 f_l^m &= \hbar^2 l(l+1) f_l^m \\ -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] f_l^m &= \hbar^2 l(l+1) f_l^m \end{aligned}$$

The eigenfunction equation for L_z is

$$L_z f_l^m = -i\hbar\frac{\partial}{\partial\phi} f_l^m = \hbar m f_l^m$$

This equation is equivalent to

$$\frac{1}{f_l^m}\frac{\partial^2 f_l^m}{\partial\phi^2} = -m^2$$

since they both have the solution $f_l^m = g(\theta)e^{im\phi}$, where $g(\theta)$ is some function of θ (the precise function doesn't matter since it cancels out on the left hand side).

Looking back at the solution to the angular part of the Schrödinger equation in three dimensions, we see that these differential equations are the same as the ones we encountered there when we used separation of variables. The solutions there were the spherical harmonics, so we see that the eigenfunctions of L^2 are in fact spherical harmonics.

One final comment is worth making. When we worked out the eigenvalues of L^2 we saw that they had the form $\hbar^2 l(l+1)$ where l could be any non-negative integer or half-integer. In the solution of the differential equations that give rise to the spherical harmonics, however, only integer values of l are allowed. You might think that these extra values of l are not physically significant, but when we come to study spin, we see that since the eigenstates of spin have no relation to spatial coordinates so that spherical harmonics are not involved, there is no reason to discard the half-integer values. In fact they turn out to be fundamental, since half-integer spins are common amongst the elementary particles (such as protons, neutrons and electrons).

PINGBACKS

Pingback: Angular momentum - eigenvalues