ANGULAR MOMENTUM - EIGENVALUES

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Sec. 4.3.1.

We saw earlier that the square of the total angular momentum commutes with all three of its components, although these components do not commute with each other. This means that L^2 and one of its components, say L_z, can be measured simultaneously. We can use this fact to work out the eigenvalues of these two commuting operators without actually working out the eigenfunctions (which turn out to be the spherical harmonics, although it takes a bit of calculation to show this).

That is, we know that we can write

$$L^2 f = \lambda f \tag{0.1}$$

$$L_z f = \mu f \tag{0.2}$$

for some eigenvalues λ and μ to be determined, but without the need to find the eigenfunction f explicitly.

To this end we define the raising and lowering operators (the reason for the names will become apparent)

$$L_\pm \equiv L_x \pm iL_y \tag{0.3}$$

Since L^2 commutes with all three components, it will also commute with $L_\pm$ so we have

$$\left[L^2, L_\pm\right] = 0 \tag{0.4}$$

The commutator $[L_z, L_\pm]$ can be worked out from the commutators of its terms, as we've seen earlier:

$$[L_x, L_y] = i\hbar L_z \tag{0.5}$$

$$[L_y, L_z] = i\hbar L_x \tag{0.6}$$

$$[L_z, L_x] = i\hbar L_y \tag{0.7}$$

Using this, we get

$$[L_z, L_\pm] = [L_z, L_x] \pm i[L_z, L_y] \tag{0.8}$$
$$= i\hbar L_y \pm i(-i\hbar L_x) \tag{0.9}$$
$$= \hbar(\pm L_x + iL_y) \tag{0.10}$$
$$= \pm\hbar(L_x \pm iL_y) \tag{0.11}$$
$$= \pm\hbar L_\pm \tag{0.12}$$

We now make the claim that if, as above, f is an eigenfunction of both L^2 and L_z, then so are the functions $L_\pm f$.

To prove this, we start with L^2:

$$L^2(L_\pm f) = L_\pm L^2 f \tag{0.13}$$
$$= L_\pm(\lambda f) \tag{0.14}$$
$$= \lambda L_\pm f \tag{0.15}$$

The result follows because $\left[L^2, L_\pm\right] = 0$. Thus f is an eigenfunction of L^2 and $L_\pm$ with the *same* eigenvalue, so applying $L_\pm$ to this eigenfunction results in states with the same square of total angular momentum. This is a crucial point which we'll use later.

Now consider L_z. We have using the commutator above

$$L_z L_\pm f = \pm\hbar L_\pm f + L_\pm L_z f \tag{0.16}$$
$$= \pm\hbar L_\pm f + L_\pm(\mu f) \tag{0.17}$$
$$= (\pm\hbar + \mu)(L_\pm f) \tag{0.18}$$

In this case, the function $L_\pm f$ has an eigenvalue that is raised or lowered by $\hbar$ depending on which operator (L_+ or L_-) is chosen. Thus we generate a sequence of functions which have a constant value of L^2 but a range of values of L_z.

Now we come to an important observation. Since L^2 is the square of the *total* angular momentum, it isn't possible for the observed value of one of its components L_z to be greater than L^2. Since applying L_+ to f generates a new eigenfunction with an eigenvalue that is larger by $\hbar$, there must come a point where this sequence of functions stops (otherwise the value of L_z would be greater than that of L^2). That is, there must be some function f_{max} such that $L_+ f_{max} = 0$.

We can assume that the eigenvalue of L_z for f_{max} is $\hbar l$ for some number l. That is, for this eigenfunction

$$L_z f_{max} = \hbar l f_{max} \tag{0.19}$$

$$L^2 f_{max} = \lambda f_{max} \tag{0.20}$$

Now we have to use one of those annoying identities that you'd never think of on your own.

$$L_\pm L_\mp = (L_x \pm iL_y)(L_x \mp iL_y) \tag{0.21}$$

$$= L_x^2 + L_y^2 \mp i(L_x L_y - L_y L_x) \tag{0.22}$$

$$= L^2 - L_z^2 \mp i(i\hbar L_z) \tag{0.23}$$

$$= L^2 - L_z^2 \pm \hbar L_z \tag{0.24}$$

$$L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z \tag{0.25}$$

The second line follows from $L^2 = L_x^2 + L_y^2 + L_z^2$ and the third line from the commutator $[L_x, L_y] = i\hbar L_z$.

Applying this operator to f_{max} and recalling that f_{max} is an eigenfunction of L^2 with eigenvalue λ and of L_z with eigenvalue $\hbar l$, we get by using the second choice of sign in each term

$$L^2 f_{max} = \left(L_- L_+ + L_z^2 + \hbar L_z\right) f_{max} \tag{0.26}$$

$$= (0 + \hbar^2 l^2 + \hbar^2 l) f_{max} \tag{0.27}$$

$$= \hbar^2 l(l+1) f_{max} \tag{0.28}$$

where the zero in the second line comes from our assumption that $L_+ f_{max} = 0$. We therefore have the eigenvalue of L^2 in terms of the maximum eigenvalue of L_z:

$$\lambda = \hbar^2 l(l+1) \tag{0.29}$$

We can apply a similar argument to the lowering operator L_-. That is, there must be an eigenfunction f_{min} with an eigenvalue $\hbar l'$ such that lowering it with L_- gives zero:

$$L_z f_{min} = \hbar l' f_{min} \tag{0.30}$$

$$L^2 f_{min} = \hbar^2 l(l+1) f_{min} \tag{0.31}$$

$$L_- f_{min} = 0 \tag{0.32}$$

The second line comes from the fact that all the eigenfunctions in this sequence have the same eigenvalue for L^2, so we can write this in terms

of the number l that we got out of the maximum eigenvalue of L_z. We can follow a similar analysis, this time taking the first choice of sign in the expression for L^2 above:

$$L^2 f_{min} = \left(L_+ L_- + L_z^2 - \hbar L_z\right) f_{min} \tag{0.33}$$

$$= (0 + (\hbar l')^2 - \hbar^2 l') f_{min} \tag{0.34}$$

$$= \hbar^2 l'(l'-1) f_{min} \tag{0.35}$$

Equating the two expressions for the eigenvalue of L^2 we get

$$l'(l'-1) = l(l+1) \tag{0.36}$$

This can be satisfied for arbitrary l if either $l' = -l$ or $l' = l+1$. The second choice doesn't work because this would make the lowest eigenvalue of L_z greater than the highest, so we must have $l' = -l$.

The conclusion of all this is that we can write the eigenvalues of L_z as $m\hbar$ where m runs from $-l$ to $+l$ in integer steps. The actual number of steps required depends on l, but since the lowest value of m is $-l$, the highest value must be $m_{max} = l = -l + N$ for some integer N. Thus $l = N/2$ so must be an integer or half-integer. This gives us the final result

$$L^2 f_l^m = \hbar^2 l(l+1) f_l^m \tag{0.37}$$

$$L_z f_l^m = \hbar m f_l^m \tag{0.38}$$

where $l = 0, 1/2, 1, 3/2, \ldots$ and for each value of l, $m = -l, -l+1, \ldots, l-1, l$.

The choice of L_z as the component we measure is arbitrary; we could just as well have chosen L_x or L_y and get the same result. The crucial point though is that we can choose only one component to measure at a time. The other two components will have indefinite values. This means that it is not correct to picture the angular momentum vector $\mathbf{L}$ as a precise vector in three-dimensional space, since doing so requires specifying all three of its components simultaneously. We can specify the magnitude of the vector (via L^2) and one of its components (such as L_z), but if we want a mental picture of what such an angular momentum would look like, we need to think of it as a vector of a fixed length with a fixed projection onto the z axis, but with the other two components smeared out in the $x-y$ plane. This is of course very difficult to do, but it's one of those things that we need to get used to in quantum mechanics.

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