WAVE FUNCTION: BORN'S CONDITIONS

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Max Born's best known contribution to quantum mechanics was his proposal that the wave function, or rather its square modulus, should be interpreted as the probability density for finding the system in a given state at a given time. However, he also proposed four conditions on the wave function which are used in finding many solutions of the Schrödinger equation. As always, it's useful to write down the Schrödinger equation (in one dimension) so we can see how Born's conditions fit in.

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+V(x,t)\Psi = i\hbar\frac{\partial\Psi}{\partial t} \tag{0.1}$$

Born's conditions to be imposed on the wave function $\Psi(x,t)$ are:

(1) The wave function must be single valued. This means that for any given values of x and t, $\Psi(x,t)$ must have a unique value. This is a way of guaranteeing that there is only a single value for the probability of the system being in a given state. Actually, if Ψ is a proper mathematical function, it will satisfy this requirement automatically, since one condition all functions must satisfy is that they are single-valued. The most common example of a 'function' (that isn't really a function) encountered by most undergraduate math students is the inverse sine (or arcsin), which gives the angle corresponding to a particular value. Thus any multiple of π has a sine of 0, so in principle, the inverse sine could give any multiple of π as its value and thus it seems this is a multi-valued function. However, in practice, only the range from 0 (inclusive) to π (exclusive) is used in the 'proper' arcsin function.

(2) The wave function must be square-integrable. In other words, the integral of $|\Psi|^2$ over all space must be finite. This is another way of saying that it must be possible to use $|\Psi|^2$ as a probability density, since any probability density must integrate over all space to give a value of 1, which is clearly not possible if the integral of $|\Psi|^2$ is infinite. One consequence of this proposal is that Ψ must tend to 0 for infinite distances.

(3) The wave function must be continuous everywhere. That is, there are no sudden jumps in the probability density when moving through

space. If a function has a discontinuity such as a sharp step upwards or downwards, this can be seen as a limiting case of a very rapid change in the function. Such a rapid change would mean that the derivative of the function was very large (either a very large positive or negative number). In the limit of a step function, this would imply an infinite derivative. Since the momentum of the system is found using the momentum operator, which is a first order derivative, this would imply an infinite momentum, which is not possible in a physically realistic system. Such an infinite derivative would also violate condition 4.

(4) All first-order derivatives of the wave function must be continuous. Following the same reasoning as in condition 3, a discontinuous first derivative would imply an infinite second derivative, and since the energy of the system is found using the second derivative, a discontinuous first derivative would imply an infinite energy, which again is not physically realistic.

Having stated Born's conditions, however, we need to note that several systems commonly studied in introductory quantum mechanics courses do violate one or more of them. For example, the 'particle in a box' system is composed of a particle moving in a box with infinitely high sides, represented by a potential function that is zero in a limited area, and infinite outside this area. In such a system, the third condition (continuity of the wave function) is imposed to find a solution, but the solution so found violates the fourth condition, in that the derivative of the wave function is *not* continuous at the boundary of the box.

The particle in a box is clearly not a physically realistic system, however, since there is no known physical mechanism which can generate an infinitely deep potential well. Despite that, the system is still useful as a model of some real-life situations in which a particle is found in a very deep well. And of course the particle in a box provides a relatively painless way of introducing many of the concepts of quantum mechanics, so is useful heuristically, even if it's not entirely realistic.

It's rather odd that most textbooks gloss over these conditions (well, they usually mention the square integrable one since it's essential for using the wave function as a probability density) by simply stating that the wave function and its derivatives are required to be continuous, without explaining why.

For a better motivation of conditions 3 and 4 see here.

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