DERIVATIVES

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Physics frequently requires the calculation of the rate of change of one variable with respect to another. (For a discussion of one such case - the notion of velocity - see here.) If, for example, we want to calculate the velocity of a mass, we can begin by measuring its position at two distinct times and then dividing the distance travelled by the time taken to travel that distance. This will give us the *average* velocity over that time interval, but it won't tell us what precise velocity the mass had at any specific time within that interval. We can get a better and better approximation to the velocity at a particular time by taking measurements at time intervals that are closer and closer together, but we are still calculating just the average velocity over a time interval, no matter how short that interval may be. If we tried to use this measurement technique for finding the velocity at once exact instant in time, we would need to take two measurements of the mass's position at the same time. These two measurements would, of course, be identical so if we attempted to calculate the velocity at that time by dividing the distance travelled by the time interval, we would have to divide zero by zero, which is mathematically forbidden.

As we showed here, the mathematical idea of the limit of a function can be used for finding the value of that function at places where such as things as division by zero occur. So what we would like to do is to define a function that gives the average speed of the mass by the usual method of dividing the distance travelled by the time interval, and then take the limit of that function as the time interval approaches zero. Hopefully, this limit will yield a finite value that we can interpret as the velocity at a specific time.

This notion of the rate of change of one quantity with respect to another is the motivation for the *derivative* of a function $f(x)$ with respect to the variable x. The derivative gives us the rate of change of a function at each individual point. Going back to the discussion about our speed as we walk along a north-south path, suppose we represent our position y on the path as a function of time: $y=y(t)$. The method we used earlier to calculate our average velocity was to measure how far we moved and then divide by the time taken to move that far. If we are at a position y_0 at time t_0 (that's our starting point) and then we move to position y_1 at time t_1, then our average speed v_{av} as we move from y_0 to y_1 is

$$v_{av} = \frac{y_1 - y_0}{t_1 - t_0} = \frac{y(t_1) - y(t_0)}{t_1 - t_0} \tag{0.1}$$

Clearly we can't use this formula directly if both times are the same ($t_1 = t_0$) since that would involve dividing zero by zero. However, as we saw in the page where we discussed limits, that is exactly the kind of problem we solved there.

We can change the notation slightly so we have a better grasp of what we're trying to do. If we take t_0 as the reference time, then we can write the later timet_1 as$t_1 = t_0 + \Delta t$ and the speed formula becomes

$$v_{av} = \frac{y(t_0 + \Delta t) - y(t_0)}{\Delta t} \tag{0.2}$$

Now if we can calculate the limit as the time interval Δt goes to zero, we should get what we're looking for: a formula that gives us the speed at one specific point, rather than an average speed over a finite time interval.

In order to proceed further, we would have to know the function $y(t)$, but we have the general idea of a derivative here. The derivative of a function $y(t)$ is the rate of change with respect to t of the function at each point (provided the original function is defined at that point). The usual notation for the derivative is:

$$\frac{dy}{dt} \equiv \lim_{\Delta t \to 0} \frac{y(t + \Delta t) - y(t)}{\Delta t} \tag{0.3}$$

When you study differential calculus, you will encounter formulas for finding the derivatives of all the standard functions such as polynomials, exponentials, trigonometric functions, logarithms and so forth. However, all of these derivatives satisfy the definition we have just given.

As an example, the page on which kinetic energy is discussed shows that if a constant force F acts on a mass located at the origin for a time t, then the mass's position at that time is

$$s = \frac{1}{2}at^2 \tag{0.4}$$

where a is the acceleration produced by the mass (which is just F/m from Newton's law). We also showed on that page that the mass's velocity at time t is

$$v = at \tag{0.5}$$

Now, since we know that velocity is rate of change of position, then if we calculate the derivative of s with respect to time t, we should get the equation for the velocity. Those who have studied calculus before will see that this is true already, but let's apply the definition of the derivative above and see if we can prove it.

We want to calculate

$$\frac{ds}{dt} = \lim_{\Delta t\to 0}\frac{s(t+\Delta t)-s(t)}{\Delta t} \tag{0.6}$$

$$= \lim_{\Delta t\to 0}\frac{a(t+\Delta t)^2/2-at^2/2}{\Delta t} \tag{0.7}$$

$$= \frac{a}{2}\lim_{\Delta t\to 0}\frac{t^2+2t\Delta t+(\Delta t)^2-t^2}{\Delta t} \tag{0.8}$$

$$= \frac{a}{2}\lim_{\Delta t\to 0}\frac{2t\Delta t+(\Delta t)^2}{\Delta t} \tag{0.9}$$

$$= \frac{a}{2}\lim_{\Delta t\to 0}(2t+\Delta t) \tag{0.10}$$

$$= at \tag{0.11}$$

Thus we see that $ds/dt = at = v$ as we hoped. Note that we are able to take the factor of $a/2$ outside the limit, since a is a constant here.

The key to calculating formulas for derivatives is to transform the expression of which you want the limit to a form where you're not dividing by zero, and we managed to achieve this above in the second-to-last line, where we were able to cancel the factor of Δt off the numerator and denominator.

There are formulas which can be derived in similar ways for all the standard algebraic, logarithmic, exponential and trigonometric functions, as well as standard formulas for products and quotients of functions and for compound functions (functions of other functions). However, what is important to us here is that the derivative is the answer to the physics problem of how to calculate rates of change at specific points.

PINGBACKS

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