GAUSSIAN DISTRIBUTION

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 1.3.

Probably the most common continuous probability density is the gaussian distribution, specified by

$$\rho(x) = Ae^{-\lambda(x-a)^2} \tag{1}$$

First, we need to normalize the distribution by finding A. That is, we must have

$$A\int_{-\infty}^{\infty} e^{-\lambda(x-a)^2} dx = 1 \tag{2}$$

The gaussian integral is very common, and the result is that

$$\int_{-\infty}^{\infty} e^{-\lambda(x-a)^2} dx = \sqrt{\frac{\pi}{\lambda}} \tag{3}$$

$$A = \sqrt{\frac{\lambda}{\pi}} \tag{4}$$

Although there is no closed form indefinite integral, the definite integral can be found by a cute trick.

$$\left[\int_{-\infty}^{\infty} e^{-x^2} dx\right]^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy \tag{5}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} dx\, dy \tag{6}$$

We can now transform to polar coordinates using

$$
\begin{aligned}
r^2 &= x^2+y^2 && (7)\\
dx\,dy &= r\,dr\,d\theta && (8)\\
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}dx\,dy &= \int_0^{\infty}\int_0^{2\pi} re^{-r^2}d\theta\,dr && (9)\\
&= 2\pi\int_0^{\infty} re^{-r^2}dr && (10)\\
&= -\pi\, e^{-r^2}\Big|_0^{\infty} && (11)\\
&= \pi && (12)
\end{aligned}
$$

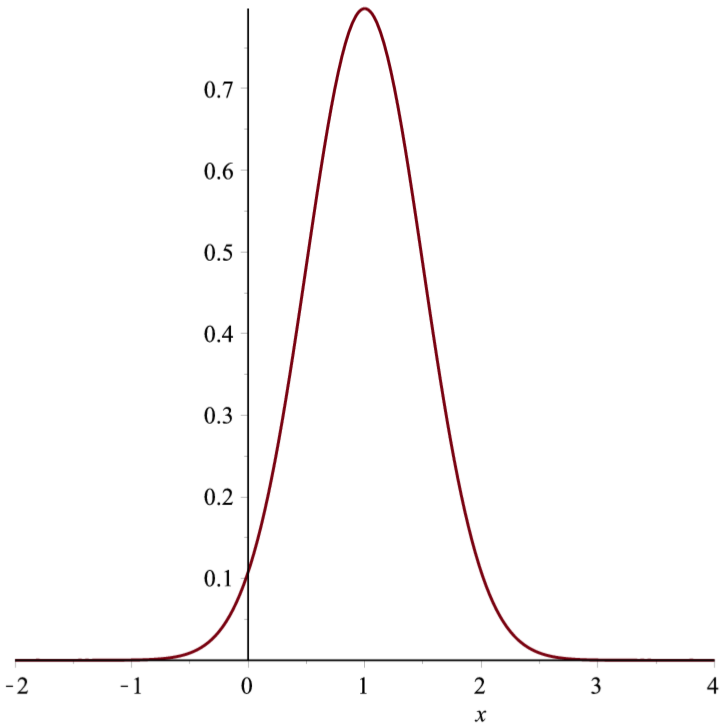
Therefore

$$
\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi} \qquad (13)
$$

Using Maple (or integration by parts) we can work out the average and variance.

$$
\begin{aligned}
\langle x\rangle &= \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} xe^{-\lambda(x-a)^2}dx = a && (14)\\
\left\langle x^2\right\rangle &= \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} x^2e^{-\lambda(x-a)^2}dx = \frac{1}{2\lambda}+a^2 && (15)\\
\sigma^2 &= \left\langle x^2\right\rangle - \langle x\rangle^2 = \frac{1}{2\lambda} && (16)
\end{aligned}
$$

The distribution has the standard bell shape. Here's a plot for $\lambda = 2$ and $a = 1$:


0.7
0.6
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