GAUSSIAN DISTRIBUTION

Link to: physicspages home page.
To leave a comment or report an error, please use the auxiliary blog.
Post date: 22 Jun 2015.

Probably the most common continuous probability density is the gaussian distribution, specified by

\[
\rho(x) = Ae^{-\lambda(x-a)^2} \quad (1)
\]

First, we need to normalize the distribution by finding \( A \). That is, we must have

\[
A \int_{-\infty}^{\infty} e^{-\lambda(x-a)^2} dx = 1 \quad (2)
\]

The gaussian integral is very common, and the result is that

\[
\int_{-\infty}^{\infty} e^{-\lambda(x-a)^2} dx = \sqrt{\frac{\pi}{\lambda}} \quad (3)
\]

\[
A = \sqrt{\frac{\lambda}{\pi}} \quad (4)
\]

Although there is no closed form indefinite integral, the definite integral can be found by a cute trick.

\[
\left[ \int_{-\infty}^{\infty} e^{-x^2} dx \right]^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy \quad (5)
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dx \, dy \quad (6)
\]

We can now transform to polar coordinates using
\( r^2 = x^2 + y^2 \) \hspace{1cm} (7)
\( dx\,dy = r\,dr\,d\theta \) \hspace{1cm} (8)
\( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} \, dx \, dy = \int_{0}^{\infty} \int_{0}^{2\pi} r e^{-r^2} \, d\theta \, dr \) \hspace{1cm} (9)
\( = 2\pi \int_{0}^{\infty} r e^{-r^2} \, dr \) \hspace{1cm} (10)
\( = -\pi e^{-r^2} \bigg|_{0}^{\infty} \) \hspace{1cm} (11)
\( = \pi \) \hspace{1cm} (12)

Therefore

\( \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \) \hspace{1cm} (13)

Using Maple (or integration by parts) we can work out the average and variance.

\[ \langle x \rangle = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x e^{-\lambda(x-a)^2} \, dx = a \] \hspace{1cm} (14)

\[ \langle x^2 \rangle = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x^2 e^{-\lambda(x-a)^2} \, dx = \frac{1}{2\lambda} + a^2 \] \hspace{1cm} (15)

\[ \sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2\lambda} \] \hspace{1cm} (16)

The distribution has the standard bell shape. Here’s a plot for \( \lambda = 2 \) and \( a = 1 \):