

GAUSSIAN DISTRIBUTION

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 1.3.

Probably the most common continuous probability density is the gaussian distribution, specified by

$$(0.1) \quad \rho(x) = Ae^{-\lambda(x-a)^2}$$

First, we need to normalize the distribution by finding A . That is, we must have

$$(0.2) \quad A \int_{-\infty}^{\infty} e^{-\lambda(x-a)^2} dx = 1$$

The gaussian integral is very common, and the result is that

$$(0.3) \quad \int_{-\infty}^{\infty} e^{-\lambda(x-a)^2} dx = \sqrt{\frac{\pi}{\lambda}}$$

$$(0.4) \quad A = \sqrt{\frac{\lambda}{\pi}}$$

Although there is no closed form indefinite integral, the definite integral can be found by a cute trick.

$$(0.5) \quad \left[\int_{-\infty}^{\infty} e^{-x^2} dx \right]^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$(0.6) \quad = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dx dy$$

We can now transform to polar coordinates using

$$(0.7) \quad r^2 = x^2 + y^2$$

$$(0.8) \quad dx dy = r dr d\theta$$

$$(0.9) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dx dy = \int_0^{\infty} \int_0^{2\pi} r e^{-r^2} d\theta dr$$

$$(0.10) \quad = 2\pi \int_0^{\infty} r e^{-r^2} dr$$

$$(0.11) \quad = -\pi e^{-r^2} \Big|_0^{\infty}$$

$$(0.12) \quad = \pi$$

Therefore

$$(0.13) \quad \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Using Maple (or integration by parts) we can work out the average and variance.

$$(0.14) \quad \langle x \rangle = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x e^{-\lambda(x-a)^2} dx = a$$

$$(0.15) \quad \langle x^2 \rangle = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x^2 e^{-\lambda(x-a)^2} dx = \frac{1}{2\lambda} + a^2$$

$$(0.16) \quad \sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2\lambda}$$

The distribution has the standard bell shape. Here's a plot for $\lambda = 2$ and $a = 1$:

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