ADDING A CONSTANT TO THE POTENTIAL INTRODUCES A PHASE FACTOR

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 1.8.

The time-independent Schrödinger equation in one dimension can be separated into two equations as follows:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+V(x)\psi(x) = E\psi(x) \tag{1}$$

$$i\hbar\frac{d\Xi(t)}{dt} = E\Xi(t) \tag{2}$$

and the general solution is

$$\Psi(x,t)=\psi(x)\,\Xi(t) \tag{3}$$

The time component can be solved as

$$\Xi(t)=Ce^{-iEt/\hbar} \tag{4}$$

where C is the constant of integration.

If we add a constant (in both space and time) V_0 to the potential, then the original Schrödinger equation becomes

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2}+V(x)\Psi+V_0\Psi = i\hbar\frac{\partial\Psi}{\partial t} \tag{5}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2}+V(x)\Psi = i\hbar\frac{\partial\Psi}{\partial t}-V_0\Psi \tag{6}$$

Applying separation of variables gives us

$$-\frac{\hbar^2}{2m}\frac{1}{\psi(x)}\frac{\partial^2\psi(x)}{\partial x^2}+V(x) = E \tag{7}$$

$$i\hbar\frac{1}{\Xi(t)}\frac{\partial\Xi}{\partial t}-V_0 = E \tag{8}$$

[Since V_0 is independent of both x and t, we could put it in either the $\psi(x)$ or the $\Xi(t)$ equation, but putting it in the Ξ equation eliminates it from the more complex ψ equation, so we'll do that.]

The solution to 8 is now

$$\Xi(t) = Ce^{-i(E+V_0)t/\hbar} \tag{9}$$

so we've introduced a phase factor $e^{-iV_0 t/\hbar}$ into the overall wave function Ψ. For the time-independent Schrödinger equation, all quantities of physical interest involve multiplying the complex conjugate Ψ^* by some operator $\hat{Q}(x)$ that depends only on x, operating on Ψ. That is, we're interested only in quantities of the form

$$\begin{aligned}
\Psi^*\left[\hat{Q}(x)\Psi\right] &= |C|^2 e^{+i(E+V_0)t/\hbar} e^{-i(E+V_0)t/\hbar} \psi^*\left[\hat{Q}(x)\psi\right] &\quad (10)\\
&= |C|^2 \psi^*\left[\hat{Q}(x)\psi\right] &\quad (11)
\end{aligned}$$

Thus the phase factor disappears when calculating any physical quantity.