ADDING A CONSTANT TO THE POTENTIAL INTRODUCES A PHASE FACTOR

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The time-independent Schrödinger equation in one dimension can be separated into two equations as follows:

\[
-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + V(x) \psi(x) = E \psi(x) \tag{1}
\]

\[
i \hbar \frac{d \Xi(t)}{dt} = E \Xi(t) \tag{2}
\]

and the general solution is

\[
\Psi(x,t) = \psi(x) \Xi(t) \tag{3}
\]

The time component can be solved as

\[
\Xi(t) = Ce^{-iEt/\hbar} \tag{4}
\]

where \(C\) is the constant of integration.

If we add a constant (in both space and time) \(V_0\) to the potential, then the original Schrödinger equation becomes

\[
-\frac{\hbar^2}{2m} \frac{d^2 \Psi}{dx^2} + V(x) \Psi + V_0 \Psi = i\hbar \frac{\partial \Psi}{\partial t} \tag{5}
\]

\[
-\frac{\hbar^2}{2m} \frac{d^2 \Psi}{dx^2} + V(x) \Psi = i\hbar \frac{\partial \Psi}{\partial t} - V_0 \Psi \tag{6}
\]

Applying separation of variables gives us

\[
-\frac{\hbar^2}{2m} \frac{1}{\psi(x)} \frac{\partial^2 \psi(x)}{\partial x^2} + V(x) = E \tag{7}
\]

\[
i \hbar \frac{1}{\Xi(t)} \frac{\partial \Xi}{\partial t} - V_0 = E \tag{8}
\]
[Since $V_0$ is independent of both $x$ and $t$, we could put it in either the $\psi(x)$ or the $\Xi(t)$ equation, but putting it in the $\Xi$ equation eliminates it from the more complex $\psi$ equation, so we’ll do that.]

The solution to $\Xi$ is now

$$\Xi(t) = Ce^{-i(E+V_0)t/\hbar}$$

so we’ve introduced a phase factor $e^{-iV_0t/\hbar}$ into the overall wave function $\Psi$. For the time-independent Schrödinger equation, all quantities of physical interest involve multiplying the complex conjugate $\Psi^*$ by some operator $\hat{Q}(x)$ that depends only on $x$, operating on $\Psi$. That is, we’re interested only in quantities of the form

$$\Psi^* \left[ \hat{Q}(x) \Psi \right] = |C|^2 e^{+i(E+V_0)t/\hbar} e^{-i(E+V_0)t/\hbar} \psi^* \left[ \hat{Q}(x) \psi \right]$$

$$= |C|^2 \psi^* \left[ \hat{Q}(x) \psi \right]$$

Thus the phase factor disappears when calculating any physical quantity.