HARMONIC OSCILLATOR: STATISTICS

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Suppose a particle is in the quantum state $\Psi(x,t) = A e^{-a m x^2/\hbar} e^{-i a t}$ (1)

where $A$ is the normalization constant and $a$ is a constant with dimensions of 1/time. We can find $A$ from normalization:

$$\int_{-\infty}^{\infty} |\Psi|^2 \, dx = 1$$
(2)

$$= |A|^2 \int_{-\infty}^{\infty} e^{-2a m x^2/\hbar} \, dx$$
(3)

$$= |A|^2 \sqrt{\frac{\pi \hbar}{2ma}}$$
(4)

$$A = \left( \frac{2ma}{\pi \hbar} \right)^{1/4}$$
(5)

The spatial component of the wave function is

$$\psi(x) = \left( \frac{2ma}{\pi \hbar} \right)^{1/4} e^{-a m x^2/\hbar}$$
(6)

and it must satisfy the time-independent Schrödinger equation in one dimension

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$
(7)

The energy $E$ can be found from the time equation:

$$i\hbar \frac{\partial \Xi}{\partial t} = E\Xi$$
(8)

where
$\Xi(t) = e^{-iat}$

Therefore

$$E = \hbar a$$

From 7 we have

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) = \left(\frac{2ma}{\pi \hbar}\right)^{1/4} a \left(\hbar - 2amx^2\right) e^{-amx^2/\hbar}$$

$$V(x) = \frac{E\psi(x) + \frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2}}{\psi(x)} = 2ma^2x^2$$

This is the harmonic oscillator potential, and the wave function is actually the ground state of that potential.

We can work out a few average values:

$$\langle x \rangle = 0$$

since $\psi(x)$ is even.

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \psi^2 dx = \frac{\hbar}{4am}$$

$$\langle p \rangle = -i\hbar \int_{-\infty}^{\infty} \psi \frac{\partial \psi}{\partial x} dx = 0$$

$$\langle p^2 \rangle = -\hbar^2 \int_{-\infty}^{\infty} \psi \frac{\partial^2 \psi}{\partial x^2} dx = \hbar ma$$

The standard deviations are

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{1}{2} \sqrt{\frac{\hbar}{ma}}$$

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\hbar ma}$$

and the uncertainty principle is

$$\sigma_x \sigma_p = \frac{\hbar}{2}$$

so in this case, the uncertainty is the minimum possible.