

HARMONIC OSCILLATOR: STATISTICS

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 1.9.

Suppose a particle is in the quantum state

$$(0.1) \quad \Psi(x, t) = A e^{-amx^2/\hbar} e^{-iat}$$

where A is the normalization constant and a is a constant with dimensions of 1/time. We can find A from normalization:

$$(0.2) \quad \int_{-\infty}^{\infty} |\Psi|^2 dx = 1$$

$$(0.3) \quad = |A|^2 \int_{-\infty}^{\infty} e^{-2amx^2/\hbar} dx$$

$$(0.4) \quad = |A|^2 \sqrt{\frac{\pi\hbar}{2ma}}$$

$$(0.5) \quad A = \left(\frac{2ma}{\pi\hbar} \right)^{1/4}$$

The spatial component of the wave function is

$$(0.6) \quad \psi(x) = \left(\frac{2ma}{\pi\hbar} \right)^{1/4} e^{-amx^2/\hbar}$$

and it must satisfy the time-independent Schrödinger equation in one dimension

$$(0.7) \quad -\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

The energy E can be found from the time equation:

$$(0.8) \quad i\hbar \frac{\partial \Xi}{\partial t} = E\Xi$$

where

$$(0.9) \quad \Xi(t) = e^{-iat}$$

Therefore

$$(0.10) \quad E = \hbar a$$

From 0.7 we have

$$(0.11) \quad -\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} = \left(\frac{2ma}{\pi\hbar}\right)^{1/4} a (\hbar - 2amx^2) e^{-amx^2/\hbar}$$

$$(0.12) \quad V(x) = \frac{E\psi(x) + \frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2}}{\psi(x)}$$

$$(0.13) \quad = 2ma^2x^2$$

This is the harmonic oscillator potential, and the wave function is actually the ground state of that potential.

We can work out a few average values:

$$(0.14) \quad \langle x \rangle = 0$$

since $\psi(x)$ is even.

$$(0.15) \quad \langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \psi^2 dx = \frac{\hbar}{4am}$$

$$(0.16) \quad \langle p \rangle = -i\hbar \int_{-\infty}^{\infty} \psi \frac{\partial \psi}{\partial x} dx = 0$$

$$(0.17) \quad \langle p^2 \rangle = -\hbar^2 \int_{-\infty}^{\infty} \psi \frac{\partial^2 \psi}{\partial x^2} dx = \hbar ma$$

The standard deviations are

$$(0.18) \quad \sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{1}{2} \sqrt{\frac{\hbar}{ma}}$$

$$(0.19) \quad \sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\hbar ma}$$

and the uncertainty principle is

$$(0.20) \quad \sigma_x \sigma_p = \frac{\hbar}{2}$$

so in this case, the uncertainty is the minimum possible.