HARMONIC OSCILLATOR: STATISTICS

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 1.9.

Suppose a particle is in the quantum state

$$\Psi(x,t) = Ae^{-amx^2/\hbar}e^{-iat} \tag{1}$$

where A is the normalization constant and a is a constant with dimensions of 1/time. We can find A from normalization:

$$\begin{aligned}
\int_{-\infty}^{\infty} |\Psi|^2 dx &= 1 && (2)\\
&= |A|^2 \int_{-\infty}^{\infty} e^{-2amx^2/\hbar} dx && (3)\\
&= |A|^2 \sqrt{\frac{\pi\hbar}{2ma}} && (4)\\
A &= \left(\frac{2ma}{\pi\hbar}\right)^{1/4} && (5)
\end{aligned}$$

The spatial component of the wave function is

$$\psi(x) = \left(\frac{2ma}{\pi\hbar}\right)^{1/4} e^{-amx^2/\hbar} \tag{6}$$

and it must satisfy the time-independent Schrödinger equation in one dimension

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x) \tag{7}$$

The energy E can be found from the time equation:

$$i\hbar\frac{\partial\Xi}{\partial t} = E\Xi \tag{8}$$

where

$$\Xi(t) = e^{-iat} \tag{9}$$

Therefore

$$E = \hbar a \tag{10}$$

From 7 we have

$$\begin{aligned}
-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} &= \left(\frac{2ma}{\pi\hbar}\right)^{1/4} a\left(\hbar - 2amx^2\right) e^{-amx^2/\hbar} && (11)\\
V(x) &= \frac{E\psi(x) + \frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}}{\psi(x)} && (12)\\
&= 2ma^2x^2 && (13)
\end{aligned}$$

This is the harmonic oscillator potential, and the wave function is actually the ground state of that potential.

We can work out a few average values:

$$\langle x \rangle = 0 \tag{14}$$

since $\psi(x)$ is even.

$$\begin{aligned}
\left\langle x^2 \right\rangle &= \int_{-\infty}^{\infty} x^2\psi^2 dx = \frac{\hbar}{4am} && (15)\\
\langle p \rangle &= -i\hbar\int_{-\infty}^{\infty}\psi\frac{\partial\psi}{\partial x}dx = 0 && (16)\\
\left\langle p^2 \right\rangle &= -\hbar^2\int_{-\infty}^{\infty}\psi\frac{\partial^2\psi}{\partial x^2}dx = \hbar ma && (17)
\end{aligned}$$

The standard deviations are

$$\begin{aligned}
\sigma_x &= \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \frac{1}{2}\sqrt{\frac{\hbar}{ma}} && (18)\\
\sigma_p &= \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\hbar ma} && (19)
\end{aligned}$$

and the uncertainty principle is

$$\sigma_x\sigma_p = \frac{\hbar}{2} \tag{20}$$

so in this case, the uncertainty is the minimum possible.