BUFFON'S NEEDLE: ESTIMATING PI

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 1.13.

This interesting little problem serves well to illustrate the notion of a probability density and its application to an experiment which can be done at home. It is known as Buffon's needle, since it is believed that Georges-Louis Leclerc, Comte de Buffon, first posed the problem in the 18th century.

Suppose we have a needle of length l and we drop this needle onto a sheet of paper on which there are a number of parallel lines spaced a distance l apart. What is the probability that the dropped needle will cross a line?

One way of analyzing this problem is to begin by considering the needle as the hypotenuse of a right triangle, with sides parallel and perpendicular to the parallel lines. Then if the needle makes an angle θ with the lines, the sides of this triangle are $l\cos\theta$ for the parallel side and $l\sin\theta$ for the perpendicular side. The side of the triangle parallel to the lines is of no interest here; what we are interested in the perpendicular component.

To see this, suppose we placed a needle of length $x<l$ on the paper in such a way that it was perpendicular to the lines. Such a needle would cover a fraction x/l of the distance between two adjacent lines. Thus the probability that a line would cross this specially dropped needle is just x/l. (Note that if $x=l$ the probability is 1, since such a needle covers the entire distance between the lines.)

Therefore, the probability that a needle that makes an angle θ with the lines crosses a line is $(l\sin\theta)/l=\sin\theta$.

How likely is the needle to drop at an angle θ? Since θ is a continuous variable, we need a probability density, rather than just a simple probability. Assuming that the needle is equally likely to drop at any angle between 0 and π ($180°$), the density must be a constant, and must integrate to 1 over the range of valid θ, so it must be $\rho(\theta)=\frac{1}{\pi}$.

The probability $P(\text{crosses line})$ that a randomly dropped needle crosses a line is therefore

$$\begin{aligned} P(\text{crosses line}) &= \int_0^{\pi} P(\text{angle }\theta\text{ crosses line})\rho(\theta)d\theta & (1)\\ &= \frac{1}{\pi}\int_0^{\pi}\sin\theta\;d\theta & (2)\\ &= \frac{2}{\pi} & (3)\\ &\approx 0.6366 & (4)\end{aligned}$$

Besides being another of those curious situations where π pops up unexpectedly, this result offers the possibility of an interesting way of whiling away a rainy afternoon. Simply by dropping a needle repeatedly onto lined paper, you can do an experiment that will determine the value of π, since

$$\begin{aligned} \pi &= \frac{2}{P(\text{crosses line})} & (5)\\ &= \frac{2}{(\text{Fraction of needles crossing lines})} & (6)\\ &= \frac{2(\text{Number of needles dropped})}{\text{Number of needles crossing lines}} & (7)\end{aligned}$$

OK, you would need to drop a lot of needles to get a decent value of π, but it's interesting that there is such a simple method for getting even an approximate value of π without using circles, triangles, angles, measurement or anything more than just counting.

PINGBACKS

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