PROBABILITY CURRENT

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 1.14.

Using the Schrödinger equation we can derive an interesting quantity called the *probability current*. Using the probabilistic interpretation of the wave function, the probability of a particle being between $x=a$ and $x=b$ is

$$P_{ab}=\int_a^b |\Psi^*\Psi|\,dx \tag{0.1}$$

The rate of change of this probability can then be expressed in terms of spatial derivatives using the Schrödinger equation:

$$\begin{aligned}
\frac{dP_{ab}}{dt} &= \int_a^b \left[\frac{\partial\Psi^*}{\partial t}\Psi+\frac{\partial\Psi}{\partial t}\Psi^*\right]dx && (0.2)\\
&= \int_a^b \left\{-\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2}-\frac{1}{i\hbar}V\Psi^*\right\}\Psi dx && (0.3)\\
&\quad +\int_a^b \left\{\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2}+\frac{1}{i\hbar}V\Psi\right\}\Psi^* dx && (0.4)\\
&= \frac{i\hbar}{2m}\int_a^b \left[\frac{\partial^2\Psi}{\partial x^2}\Psi^*-\frac{\partial^2\Psi^*}{\partial x^2}\Psi\right]dx && (0.5)
\end{aligned}$$

We can now apply integration by parts to each term.

$$\begin{aligned}
\int_a^b \frac{\partial^2\Psi}{\partial x^2}\Psi^* dx &= \left.\frac{\partial\Psi}{\partial x}\Psi^*\right|_a^b-\int_a^b\frac{\partial\Psi}{\partial x}\frac{\partial\Psi^*}{\partial x}dx && (0.6)\\
-\int_a^b \frac{\partial^2\Psi^*}{\partial x^2}\Psi dx &= -\left.\frac{\partial\Psi^*}{\partial x}\Psi\right|_a^b+\int_a^b\frac{\partial\Psi}{\partial x}\frac{\partial\Psi^*}{\partial x}dx && (0.7)
\end{aligned}$$

Adding these terms together, we get

$$\frac{dP_{ab}}{dt}=\frac{i\hbar}{2m}\left[\left.\frac{\partial\Psi}{\partial x}\Psi^*\right|_a^b-\left.\frac{\partial\Psi^*}{\partial x}\Psi\right|_a^b\right] \tag{0.8}$$

If we define the probability current as

$$J(x,t) \equiv \frac{i\hbar}{2m}\left(\frac{\partial\Psi^*}{\partial x}\Psi - \frac{\partial\Psi}{\partial x}\Psi^*\right) \tag{0.9}$$

we can write the rate of change of probability as

$$\frac{dP_{ab}}{dt} = J(a,t) - J(b,t) \tag{0.10}$$

As an example, if the wave function is given by

$$\Psi(x,t) = \left(\frac{2am}{\pi\hbar}\right)^{1/4} e^{-a\left[\left(mx^2/\hbar\right)+it\right]} \tag{0.11}$$

(we've taken the constant in front so that it normalizes the wave function), then

$$\frac{\partial\Psi}{\partial x} = -\frac{1}{\pi^{1/4}}\left(\frac{2am}{\hbar}\right)^{5/4} xe^{-a\left[\left(mx^2/\hbar\right)+it\right]} \tag{0.12}$$

So for the probability current we get

$$J(x,t) = -\frac{i\hbar}{2m}\frac{1}{\pi^{1/4}}\left(\frac{2am}{\hbar}\right)^{5/4} xe^{-a\left[\left(mx^2/\hbar\right)-it\right]}\left(\frac{2am}{\pi\hbar}\right)^{1/4} e^{-a\left[\left(mx^2/\hbar\right)+it\right]} \tag{0.13}$$

$$+\frac{i\hbar}{2m}\frac{1}{\pi^{1/4}}\left(\frac{2am}{\hbar}\right)^{5/4} xe^{-a\left[\left(mx^2/\hbar\right)+it\right]}\left(\frac{2am}{\pi\hbar}\right)^{1/4} e^{-a\left[\left(mx^2/\hbar\right)-it\right]} \tag{0.14}$$

$$= 0 \tag{0.15}$$

A bit of an anti-climax after all that mathematics.

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