

INTEGRAL OF INNER PRODUCT OF TWO WAVE FUNCTIONS IS CONSTANT IN TIME

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 1.16.

The fact that the normalization of the wave function is constant over time is actually a special case of a more general theorem, which is

$$(0.1) \quad \frac{d}{dt} \int_{-\infty}^{\infty} \Psi_1^* \Psi_2 dx = 0$$

for any two normalizable solutions to the Schrödinger equation (with the same potential). The proof of this follows a similar derivation to that in section 1.4 of Griffiths's book.

The derivative in the integrand is (where we're using a subscript t or x to denote a derivative with respect that variable):

$$(0.2) \quad \frac{\partial}{\partial t} (\Psi_1^* \Psi_2) = \Psi_{1t}^* \Psi_2 + \Psi_1^* \Psi_{2t}$$

From the Schrödinger equation

$$(0.3) \quad \Psi_{2t} = i \frac{\hbar}{2m} \Psi_{2xx} - \frac{i}{\hbar} V \Psi_2$$

$$(0.4) \quad \Psi_{1t}^* = -i \frac{\hbar}{2m} \Psi_{1xx}^* + \frac{i}{\hbar} V \Psi_1^*$$

(0.5)

$$(0.6) \quad \begin{aligned} \Psi_{1t}^* \Psi_2 + \Psi_1^* \Psi_{2t} &= i \frac{\hbar}{2m} (-\Psi_{1xx}^* \Psi_2 + \Psi_{2xx} \Psi_1^*) + \frac{i}{\hbar} V (\Psi_1^* \Psi_2 - \Psi_1^* \Psi_2) \\ &= i \frac{\hbar}{2m} \frac{\partial}{\partial x} (\Psi_{2x} \Psi_1^* - \Psi_{1x}^* \Psi_2) \end{aligned}$$

Inserting this into 0.1 and integrating gives zero because all wave functions go to zero at infinity. [Of course, the theorem doesn't hold if Ψ_1 and Ψ_2 are solutions for different potentials, because in that case the potential term wouldn't cancel out in 0.5.]