

INFINITE SQUARE WELL - MINIMUM ENERGY

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.3.

We've seen that the energy of a system must always be greater than the minimum of the potential function. As a specific example of this we can look at the Schrödinger equation for the square well, between $x = 0$ and $x = a$:

$$(1) \quad \frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}E\psi$$

If $E = 0$, $\psi'' = 0$. Integrating gives $\psi = Ax + B$. Attempting to satisfy the boundary conditions, we get $\psi(0) = 0$ giving $B = 0$. Then the condition $\psi(a) = 0$ gives $A = 0$, thus $\psi(x) = 0$ and cannot be normalized.

If $E < 0$, we solve the equation

$$(2) \quad \psi'' = -\frac{2mE}{\hbar^2}\psi \equiv k^2\psi$$

with $k = \sqrt{-2mE/\hbar^2}$. Since $E < 0$, k is real. The general solution is $\psi(x) = Ae^{kx} + Be^{-kx}$. Applying boundary conditions, we get $\psi(0) = 0 = A + B$, so $A = -B$. At $x = a$, we have

$$(3) \quad \psi(a) = 0$$

$$(4) \quad = Ae^{ka} + Be^{-ka}$$

$$(5) \quad = AC + \frac{B}{C}$$

$$(6) \quad = A \left(C - \frac{1}{C} \right)$$

where $C \equiv e^{ka} > 0$. Since C is strictly positive (being an exponential) the only way we can get $C - 1/C = 0$ is for $C = 1$, implying $ka = 0$. However, neither k nor a is zero here, so $C \neq 1$, so $A = 0 = B$ and $\psi(x) = 0$ again. Thus

if $E \leq 0$, the wave function cannot be normalized *and* satisfy the boundary conditions.