INFINITE SQUARE WELL - UNCERTAINTY PRINCIPLE

We can calculate the mean values of position and momentum and verify the uncertainty principle for the infinite square well. The Schrödinger equation for the square well is, between $x = 0$ and $x = a$:

$$\frac{d^2 \psi}{dx^2} = -\frac{2m}{\hbar^2} E\psi$$  \hspace{1cm} (1)

The stationary states of the infinite square well are given by

$$\sqrt{\frac{2}{a}} \sin \left(\frac{n\pi x}{a}\right) e^{-i(n^2\pi^2\hbar/2ma^2)t}$$  \hspace{1cm} (2)

for $0 \leq x \leq a$.

For $x$ we have

$$\langle x \rangle = \frac{2}{a} \int_0^a x \sin^2(\frac{n\pi x}{a}) dx = a/2$$  \hspace{1cm} (3)

$$\langle x^2 \rangle = \frac{2}{a} \int_0^a x^2 \sin^2(\frac{n\pi x}{a}) dx = a^2 \left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right)$$  \hspace{1cm} (4)

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 = a^2 \left(\frac{1}{12} - \frac{1}{2n^2\pi^2}\right) = \frac{n^2\pi^2 - 6}{12n^2\pi^2} a^2$$  \hspace{1cm} (5)

For the momentum $p$ we have

$$\langle p \rangle = \frac{2\hbar}{ai} \int_0^a \sin(n\pi x/a)(n\pi/a) \cos(n\pi x/a) dx = 0$$  \hspace{1cm} (6)

$$\langle p^2 \rangle = \frac{2\hbar^2}{a} \int_0^a \sin(n\pi x/a)(n\pi/a)^2 \sin(n\pi x/a) dx = \frac{n^2\pi^2\hbar^2}{a^2}$$  \hspace{1cm} (7)

$$\sigma_p^2 = \langle p^2 \rangle - \langle p \rangle^2 = \frac{n^2\pi^2\hbar^2}{a^2}$$  \hspace{1cm} (8)

The uncertainty principle here is then:
\[ \sigma_x \sigma_p = \hbar \sqrt{\frac{n^2 - n^2 - 6}{12}} \]  

The smallest uncertainty will be for the state \( n = 1 \) and is approximately \( 0.568 \hbar \), which satisfies the condition \( \sigma_x \sigma_p \geq \hbar/2 \).

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