INFINITE SQUARE WELL - AVERAGE ENERGY

The average or expectation value of the energy of a particle in an infinite square well can be worked out either by using the series solution in the form

$$\langle H \rangle = \sum_n |c_n|^2 E_n$$  \hspace{1cm} (1)

or directly using an integral, using $H = \frac{p^2}{2m}$ and $p = (\hbar/i) \frac{d}{dx}$:

$$\langle H \rangle = -\frac{\hbar^2}{2m} \int_0^a \Psi^*(x,t) \frac{d^2}{dx^2} \Psi(x,t) dx$$  \hspace{1cm} (2)

Since $\Psi(x,t)$ in the general case is a sum over stationary states, as in

$$\Psi(x,t) = \sum_n c_n \psi_n(x)e^{-iE_nt/\hbar}$$  \hspace{1cm} (3)

and the $\psi_n(x)$ functions are orthogonal, all the terms of the form

$$-\frac{\hbar^2}{2m} \int_0^a c_n^* \psi_n^*(x)e^{iE_n t/\hbar} c_m \psi_m(x)e^{-iE_m t/\hbar} dx$$  \hspace{1cm} (4)

where $m \neq n$ in the integral evaluate to zero, and the remaining terms where $n = m$ are all independent of time since the complex exponentials cancel out, so $\langle H \rangle$ is independent of time. Thus if we know the wave function at any given time, we can use it to work out $\langle H \rangle$ at all times.

For example, if we have a parabolic wave function at $t = 0$

$$\Psi(x,0) = Ax(a-x)$$  \hspace{1cm} (5)

for $0 \leq x \leq a$, we can work out $\langle H \rangle$ directly from it. Applying the normalization condition we can find $A$. 

\[ |A|^2 \int_0^a x^2(a-x)^2 \, dx = 1 \]  \hspace{1cm} (6)

\[ A = \sqrt{\frac{30}{a^5}} \]  \hspace{1cm} (7)

We get for \( \langle H \rangle \):

\[ \langle H \rangle = \frac{30}{a^5} \int_0^a x(a-x) \left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} x(a-x) \right) \, dx \]  \hspace{1cm} (8)

Working out the second derivative and then the integral (use Maple or do by hand) we get \( \langle H \rangle = 5\hbar^2/ma^2 \).