HARMONIC OSCILLATOR - THREE LOWEST STATIONARY STATES

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.10.

We've seen that the stationary states of the harmonic oscillator can be generated using the raising operator starting with the ground state. The process involves the spadework of calculating the derivative of each function to find the next one, which after the first few gets to be quite tedious.

We've seen that the ground state is

$$\psi_0=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-m\omega x^2/2\hbar} \tag{1}$$

The raising operator is

$$a_+ = \frac{1}{\sqrt{2\hbar m\omega}}\left[-ip+m\omega x\right] \tag{2}$$

$$= \frac{1}{\sqrt{2\hbar m\omega}}\left[-\hbar\frac{d}{dx}+m\omega x\right] \tag{3}$$

Applying this to the ground state we can generate the first two excited states (I cheated and used Maple to do the derivatives, and also to calculate the normalization constant by doing the integral. We get

$$\psi_1=\frac{\sqrt{2}}{\pi^{1/4}}\left(\frac{m\omega}{\hbar}\right)^{3/4}xe^{-m\omega x^2/2\hbar} \tag{4}$$

To find ψ_2 we start with ψ_1 and apply the raising operator again:

(5)

$$\psi_2(x)=A_2a_+\psi_1(x)$$

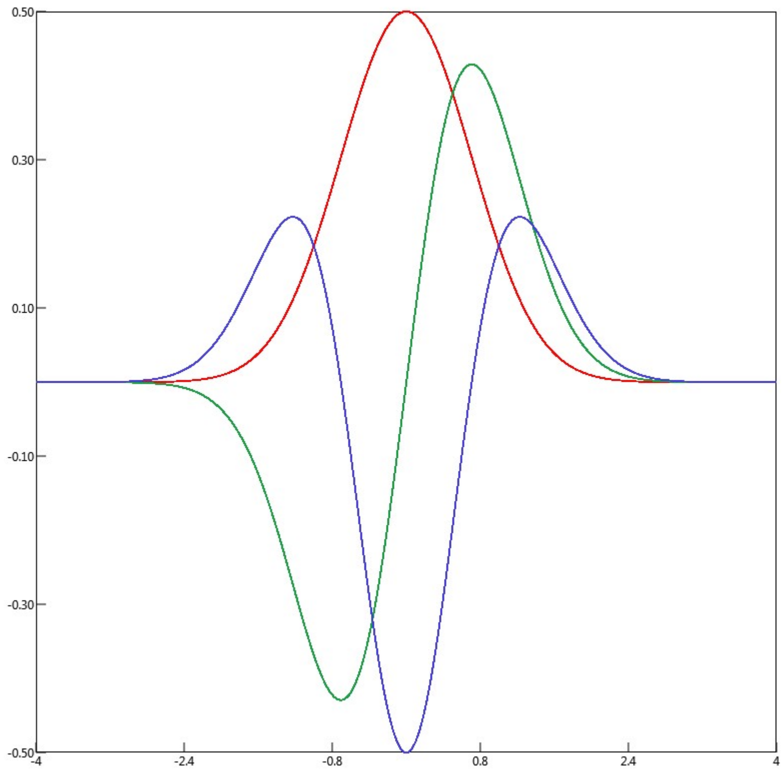
$$=A_2\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}\right)^{1/2}\frac{1}{\sqrt{2\hbar m\omega}}(-ip+m\omega x)(xe^{-m\omega x^2/2\hbar}) \tag{6}$$

$$=A_2\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(2\frac{m\omega}{\hbar}x^2-1\right)e^{-m\omega x^2/2\hbar} \tag{7}$$

where we have used $p = (h/i)\partial/\partial x$.

By normalizing this function (use Maple to do the integral), we find $A_2 = 1/\sqrt{2}$.

We can sketch the first three stationary states (not to scale) and get something like this, with ψ_0 in red, ψ_1 in green and ψ_2 in blue:



We note that since x is an odd function, and e^{-ax^2} and any constant are both even functions, and that the integral over any interval symmetric about the origin of an odd function multiplied by an even function is zero, then we can see that ψ_0 and ψ_2 are even, and ψ_1 is odd. Therefore, ψ_1 is automatically orthogonal to both ψ_0 and ψ_2 so the only thing that needs to be checked is that ψ_0 and ψ_2 are mutually orthogonal. This can be checked in Maple, by integrating with respect to x and using 'assuming positive' to make all the other constants positive. Defining $a = m\omega/2\hbar$, the integral we need to do is

$$\int_{-\infty}^{\infty} \left(4ax^2 - 1\right) e^{-2ax^2} dx \tag{8}$$

and this in fact zero.

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