

## HARMONIC OSCILLATOR - POSITION, MOMENTUM AND ENERGY

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Post date: 16 Jul 2012.

Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.11.

The lowest two states for the harmonic oscillator are the ground state  $\psi_0$  and first excited state  $\psi_1$ . We can work out some mean values of various quantities using explicit integration.

Consider first the ground state  $\psi_0$ . Using the substitutions  $\xi \equiv \sqrt{m\omega/\hbar}x$  and  $\alpha \equiv (m\omega/\pi\hbar)^{1/4}$  we have  $\psi_0 = \alpha e^{-\xi^2/2}$ . We can simplify the operations considerably if we note the even and odd natures of some of the functions to be integrated. Since  $\psi_0(x)$  is even,  $x\psi_0^2(x)$  is odd, so  $\langle x \rangle = 0$ . To calculate  $\langle p \rangle$ , since the operator  $p = (\hbar/i)d/dx$ , we need the derivative  $d\psi_0/dx = \sqrt{\hbar/m\omega}d\psi_0/d\xi = \sqrt{\hbar/m\omega}(-2\xi)\psi_0$ . This is again an odd function, so  $\langle p \rangle = 0$  as well.

For the mean square values, we do need to do some integrals (I've used software for this).

$$\begin{aligned} (1) \quad \langle x^2 \rangle &= \frac{\hbar}{m\omega} \langle \xi^2 \rangle \\ (2) \quad &= \left( \frac{\hbar}{m\omega} \right)^{3/2} \alpha^2 \int_{-\infty}^{\infty} \xi^2 e^{-\xi^2} d\xi \\ (3) \quad &= \frac{\hbar}{2m\omega} \end{aligned}$$

Here we used  $d\xi \equiv \sqrt{m\omega/\hbar}dx$  to convert the integration variable from  $x$  to  $\xi$  in the second line.

$$\begin{aligned} (4) \quad \langle p^2 \rangle &= \sqrt{m\omega\hbar}^{3/2} \alpha^2 \int_{-\infty}^{\infty} (1 - \xi^2) e^{-\xi^2} d\xi \\ (5) \quad &= \frac{1}{2} m\omega\hbar \end{aligned}$$

For  $\psi_1$  we have  $\psi_1(\xi) = \sqrt{\frac{2\pi\hbar}{m\omega}} \alpha^3 \xi e^{-\xi^2/2}$  which is an odd function. The square of an odd function is an even function, so  $\psi_1^2(\xi)$  is even, which means that the function to be integrated to find  $\langle x \rangle$  is again the product of an even function and an odd function, so  $\langle x \rangle = 0$  here as well.

Considering  $\langle p \rangle$ , we calculate the derivative of  $\psi_1(\xi)$ , which is of form  $K(1 - \xi^2)e^{-\xi^2/2}$  for a constant  $K$ , which is even. Thus to obtain  $\langle p \rangle$  we must integrate this even function multiplied by the odd function  $\psi_1$  so the result is  $\langle p \rangle = 0$ .

To get the mean square values, we do the integrals:

$$\begin{aligned} (6) \quad \langle x^2 \rangle &= \frac{\hbar}{m\omega} \langle \xi^2 \rangle \\ (7) &= 2\pi \left( \frac{\hbar}{m\omega} \right)^{5/2} \alpha^6 \int_{-\infty}^{\infty} \xi^4 e^{-\xi^2} d\xi \\ (8) &= \frac{3}{2} \frac{\hbar}{m\omega} \end{aligned}$$

And for the momentum:

$$\begin{aligned} (9) \quad \langle p^2 \rangle &= -2\pi \frac{\hbar^{5/3}}{\sqrt{m\omega}} \alpha^6 \int_{-\infty}^{\infty} \xi e^{-\xi^2/2} \frac{d^2}{d\xi^2} (\xi e^{-\xi^2/2}) d\xi \\ (10) &= \frac{3}{2} \hbar m\omega \end{aligned}$$

For  $\psi_0$  using the results above, the uncertainty principle here comes out to

$$(11) \quad \sigma_p \sigma_x = \sqrt{\langle x^2 \rangle \langle p^2 \rangle} = \frac{\hbar}{2}$$

For  $\psi_1$ , we have

$$(12) \quad \sigma_p \sigma_x = \sqrt{\langle x^2 \rangle \langle p^2 \rangle} = \frac{3\hbar}{2}$$

The mean kinetic and potential energies can be worked out from the above results without doing any more integration. We get  $\langle T \rangle = \frac{\langle p^2 \rangle}{2m} = \hbar\omega/4$  for  $\psi_0$  and  $\langle T \rangle = \frac{\langle p^2 \rangle}{2m} = 3\hbar\omega/4$  for  $\psi_1$ .

$\langle V \rangle = \frac{k\langle x^2 \rangle}{2} = \hbar\omega/4$  for  $\psi_0$  and  $\langle V \rangle = \frac{k\langle x^2 \rangle}{2} = 3\hbar\omega/4$  for  $\psi_1$ . Adding these together to get the total energy  $E$  gives  $\hbar\omega/2$  for  $\psi_0$  and  $3\hbar\omega/2$  for  $\psi_1$  as it should.

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