

HARMONIC OSCILLATOR - POSITION, MOMENTUM AND ENERGY

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.11.

The lowest two states for the harmonic oscillator are the ground state ψ_0 and first excited state ψ_1 . We can work out some mean values of various quantities using explicit integration.

Consider first the ground state ψ_0 . Using the substitutions $\xi \equiv \sqrt{m\omega/\hbar}x$ and $\alpha \equiv (m\omega/\pi\hbar)^{1/4}$ we have $\psi_0 = \alpha e^{-\xi^2/2}$. We can simplify the operations considerably if we note the even and odd natures of some of the functions to be integrated. Since $\psi_0(x)$ is even, $x\psi_0^2(x)$ is odd, so $\langle x \rangle = 0$. To calculate $\langle p \rangle$, since the operator $p = (\hbar/i)d/dx$, we need the derivative $d\psi_0/dx = \sqrt{\hbar/m\omega}d\psi_0/d\xi = \sqrt{\hbar/m\omega}(-2\xi)\psi_0$. This is again an odd function, so $\langle p \rangle = 0$ as well.

For the mean square values, we do need to do some integrals (I've used software for this).

$$\langle x^2 \rangle = \frac{\hbar}{m\omega} \langle \xi^2 \rangle \quad (1)$$

$$= \left(\frac{\hbar}{m\omega} \right)^{3/2} \alpha^2 \int_{-\infty}^{\infty} \xi^2 e^{-\xi^2} d\xi \quad (2)$$

$$= \frac{\hbar}{2m\omega} \quad (3)$$

Here we used $d\xi \equiv \sqrt{m\omega/\hbar}dx$ to convert the integration variable from x to ξ in the second line.

$$\langle p^2 \rangle = \sqrt{m\omega\hbar}^{3/2} \alpha^2 \int_{-\infty}^{\infty} (1 - \xi^2) e^{-\xi^2} d\xi \quad (4)$$

$$= \frac{1}{2} m\omega\hbar \quad (5)$$

For ψ_1 we have $\psi_1(\xi) = \sqrt{\frac{2\pi\hbar}{m\omega}} \alpha^3 \xi e^{-\xi^2/2}$ which is an odd function. The square of an odd function is an even function, so $\psi_1^2(\xi)$ is even, which means that the function to be integrated to find $\langle x \rangle$ is again the product of an even function and an odd function, so $\langle x \rangle = 0$ here as well.

Considering $\langle p \rangle$, we calculate the derivative of $\psi_1(\xi)$, which is of form $K(1 - \xi^2)e^{-\xi^2/2}$ for a constant K , which is even. Thus to obtain $\langle p \rangle$ we must integrate this even function multiplied by the odd function ψ_1 so the result is $\langle p \rangle = 0$.

To get the mean square values, we do the integrals:

$$\langle x^2 \rangle = \frac{\hbar}{m\omega} \langle \xi^2 \rangle \quad (6)$$

$$= 2\pi \left(\frac{\hbar}{m\omega} \right)^{5/2} \alpha^6 \int_{-\infty}^{\infty} \xi^4 e^{-\xi^2} d\xi \quad (7)$$

$$= \frac{3}{2} \frac{\hbar}{m\omega} \quad (8)$$

And for the momentum:

$$\langle p^2 \rangle = -2\pi \frac{\hbar^{5/3}}{\sqrt{m\omega}} \alpha^6 \int_{-\infty}^{\infty} \xi e^{-\xi^2/2} \frac{d^2}{d\xi^2} (\xi e^{-\xi^2/2}) d\xi \quad (9)$$

$$= \frac{3}{2} \hbar m \omega \quad (10)$$

For ψ_0 using the results above, the uncertainty principle here comes out to

$$\sigma_p \sigma_x = \sqrt{\langle x^2 \rangle \langle p^2 \rangle} = \frac{\hbar}{2} \quad (11)$$

For ψ_1 , we have

$$\sigma_p \sigma_x = \sqrt{\langle x^2 \rangle \langle p^2 \rangle} = \frac{3\hbar}{2} \quad (12)$$

The mean kinetic and potential energies can be worked out from the above results without doing any more integration. We get $\langle T \rangle = \frac{\langle p^2 \rangle}{2m} = \hbar\omega/4$ for ψ_0 and $\langle T \rangle = \frac{\langle p^2 \rangle}{2m} = 3\hbar\omega/4$ for ψ_1 .

$\langle V \rangle = \frac{k\langle x^2 \rangle}{2} = \hbar\omega/4$ for ψ_0 and $\langle V \rangle = \frac{k\langle x^2 \rangle}{2} = 3\hbar\omega/4$ for ψ_1 . Adding these together to get the total energy E gives $\hbar\omega/2$ for ψ_0 and $3\hbar\omega/2$ for ψ_1 as it should.

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