HARMONIC OSCILLATOR - POSITION, MOMENTUM AND ENERGY

Link to: physicspages home page.
To leave a comment or report an error, please use the auxiliary blog.
Post date: 16 Jul 2012.

The lowest two states for the harmonic oscillator are the ground state $\psi_0$ and first excited state $\psi_1$. We can work out some mean values of various quantities using explicit integration.

Consider first the ground state $\psi_0$. Using the substitutions $\xi \equiv \sqrt{m/\hbar \omega} x$ and $\alpha \equiv (m\omega/\pi \hbar)^{1/4}$ we have $\psi_0 = \alpha e^{-\xi^2/2}$. We can simplify the operations considerably if we note the even and odd natures of some of the functions to be integrated. Since $\psi_0(x)$ is even, $x\psi_0^2(x)$ is odd, so $\langle x \rangle = 0$.

To calculate $\langle p \rangle$, since the operator $p = (\hbar/i) d/dx$, we need the derivative $d\psi_0/dx = \sqrt{\hbar/m\omega} d\psi_0/d\xi = \sqrt{\hbar/m\omega} (-2\xi)\psi_0$. This is again an odd function, so $\langle p \rangle = 0$ as well.

For the mean square values, we do need to do some integrals (I’ve used software for this).

$$\langle x^2 \rangle = \frac{\hbar}{m\omega} \langle \xi^2 \rangle$$

$$= \frac{\hbar}{m\omega} \left( \frac{\hbar}{m\omega} \right)^{3/2} \alpha^2 \int_{-\infty}^{\infty} \xi^2 e^{-\xi^2} d\xi$$

$$= \frac{\hbar}{2m\omega}$$

Here we used $d\xi \equiv \sqrt{m\omega/\hbar} dx$ to convert the integration variable from $x$ to $\xi$ in the second line.

$$\langle p^2 \rangle = \sqrt{m\omega} \hbar^{3/2} \alpha^2 \int_{-\infty}^{\infty} (1-\xi^2) e^{-\xi^2} d\xi$$

$$= \frac{1}{2} m\omega \hbar$$
For \( \psi_1 \) we have \( \psi_1(\xi) = \sqrt{\frac{2\pi \hbar}{m\omega}} \alpha^3 \xi e^{-\xi^2/2} \) which is an odd function. The square of an odd function is an even function, so \( \psi_1^2(\xi) \) is even, which means that the function to be integrated to find \( \langle x \rangle \) is again the product of an even function and an odd function, so \( \langle x \rangle = 0 \) here as well.

Considering \( \langle p \rangle \), we calculate the derivative of \( \psi_1(\xi) \), which is of form \( K(1 - \xi^2)e^{-\xi^2/2} \) for a constant \( K \), which is even. Thus to obtain \( \langle p \rangle \) we must integrate this even function multiplied by the odd function \( \psi_1 \) so the result is \( \langle p \rangle = 0 \).

To get the mean square values, we do the integrals:

\[
\langle x^2 \rangle = \frac{\hbar}{m\omega} \langle \xi^2 \rangle = 2\pi \left( \frac{\hbar}{m\omega} \right)^{5/2} \alpha^6 \int_{-\infty}^{\infty} \xi^4 e^{-\xi^2} d\xi
\]

\[
= \frac{3}{2} \frac{\hbar}{m\omega}
\]

And for the momentum:

\[
\langle p^2 \rangle = -2\pi \left( \frac{\hbar}{\sqrt{m\omega}} \right)^{5/3} \alpha^6 \int_{-\infty}^{\infty} \xi e^{-\xi^2/2} \frac{d^2}{d\xi^2} (\xi e^{-\xi^2/2}) d\xi
\]

\[
= \frac{3}{2} \frac{\hbar}{m\omega}
\]

For \( \psi_0 \) using the results above, the uncertainty principle here comes out to

\[
\sigma_p \sigma_x = \sqrt{\langle x^2 \rangle \langle p^2 \rangle} = \frac{\hbar}{2}
\]

For \( \psi_1 \), we have

\[
\sigma_p \sigma_x = \sqrt{\langle x^2 \rangle \langle p^2 \rangle} = \frac{3\hbar}{2}
\]

The mean kinetic and potential energies can be worked out from the above results without doing any more integration. We get \( \langle T \rangle = \langle p^2 \rangle / 2m = \hbar \omega / 4 \) for \( \psi_0 \) and \( \langle T \rangle = \langle x^2 \rangle / 2m = 3\hbar \omega / 4 \) for \( \psi_1 \).

\( \langle V \rangle = k\langle x^2 \rangle / 2 = \hbar \omega / 4 \) for \( \psi_0 \) and \( \langle V \rangle = k\langle x^2 \rangle / 2 = 3\hbar \omega / 4 \) for \( \psi_1 \). Adding these together to get the total energy \( E \) gives \( \hbar \omega / 2 \) for \( \psi_0 \) and \( 3\hbar \omega / 2 \) for \( \psi_1 \) as it should.
PINGBACKS

Pingback: Virial theorem
Pingback: Exchange force: harmonic oscillator
Pingback: Perturbing the 3-d harmonic oscillator