

HARMONIC OSCILLATOR - RAISING AND LOWERING OPERATOR CALCULATIONS

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.12.

Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 7.4, Exercise 7.4.2.

In the study of the harmonic oscillator, we can express x and p in terms of the raising and lowering operators:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \quad (1)$$

$$p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-) \quad (2)$$

We now have

$$\langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}} \int \psi_n^*(a_+ + a_-)\psi_n dx \quad (3)$$

$$= 0 \quad (4)$$

The reason this is zero is that, as we saw when working out the normalization of the stationary states,

$$a_+ \psi_n = \sqrt{n+1} \psi_{n+1} \quad (5)$$

$$a_- \psi_n = \sqrt{n} \psi_{n-1} \quad (6)$$

$$a_+ a_- \psi_n = n \psi_n \quad (7)$$

$$a_- a_+ \psi_n = (n+1) \psi_n \quad (8)$$

and since the wave functions are orthogonal, we get

$$\int \psi_n^* \psi_{n+1} dx = \int \psi_n^* \psi_{n-1} dx = 0 \quad (9)$$

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Similarly:

$$\langle p \rangle = i\sqrt{\frac{\hbar m \omega}{2}} \int \psi_n^* (a_+ - a_-) \psi_n dx \quad (10)$$

$$= 0 \quad (11)$$

for the same reason.

For the mean squares:

$$\langle x^2 \rangle = \left(\frac{\hbar}{2m\omega} \right) \int \psi_n^* (a_+ + a_-)(a_+ + a_-) \psi_n dx \quad (12)$$

$$= \left(\frac{\hbar}{2m\omega} \right) \int \psi_n^* (a_+ a_- + a_- a_+) \psi_n dx \quad (13)$$

$$= \left(\frac{\hbar}{2m\omega} \right) (2n + 1) \quad (14)$$

$$= \frac{\hbar}{m\omega} \left(n + \frac{1}{2} \right) \quad (15)$$

In going from the first to the second line, we've thrown out terms where we integrate two orthogonal functions. For example,

$$\int \psi_n^* a_+ a_+ \psi_n dx = \int \psi_n^* \sqrt{(n+1)(n+2)} \psi_{n+2} dx \quad (16)$$

$$= 0 \quad (17)$$

We have used the relations above and the fact that ψ_n is normalized to get the third line.

Similarly:

$$\langle p^2 \rangle = -\frac{\hbar m \omega}{2} \int \psi_n^* (-a_+ a_- - a_- a_+) \psi_n dx \quad (18)$$

$$= \hbar m \omega \left(n + \frac{1}{2} \right) \quad (19)$$

The uncertainty principle then becomes

$$\sigma_p \sigma_x = \sqrt{\langle x^2 \rangle \langle p^2 \rangle} \quad (20)$$

$$= \hbar \left(n + \frac{1}{2} \right) \quad (21)$$

and the kinetic energy is

$$\langle T \rangle = \frac{\langle p^2 \rangle}{2m} = \frac{1}{2} \hbar \omega \left(n + \frac{1}{2} \right) \quad (22)$$

which is half the total energy, as it should be.

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