HARMONIC OSCILLATOR - MIXED INITIAL STATE

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.13.

As another example of the use of raising and lowering operators, consider a harmonic oscillator in the initial state of

$$\Psi(x,0)=A(3\psi_0+4\psi_1) \tag{1}$$

Normalizing, we get

$$\begin{aligned}\int_{-\infty}^{\infty}|\Psi(x,0)|^2\,dx &= |A|^2\left(3^2+4^2\right) &(2)\\ &= 1 &(3)\\ A &= \frac{1}{5} &(4)\end{aligned}$$

The time-dependent wave function is therefore

$$\Psi(x,t)=\frac{1}{5}(3\psi_0(x)e^{-iE_0t/\hbar}+4\psi_1(x)e^{-iE_1t/\hbar}) \tag{5}$$

Since

$$\begin{aligned}E_0 &= \frac{\hbar\omega}{2} &(6)\\ E_1 &= \frac{3\hbar\omega}{2} &(7)\end{aligned}$$

we can construct the probability function

$$\begin{aligned}|\Psi(x,t)|^2 &= \frac{1}{25}\left(9\psi_0^2+16\psi_1^2+12\psi_0\psi_1\left(e^{i\omega t}+e^{-i\omega t}\right)\right) &(8)\\ &= \frac{1}{25}(9\psi_0^2+16\psi_1^2+24\psi_0\psi_1\cos(\omega t)) &(9)\end{aligned}$$

The mean position $\langle x\rangle$ is most easily calculated by expressing x in terms of the raising and lowering operators:

$$x=\sqrt{\frac{\hbar}{2m\omega}}(a_{+}+a_{-}) \tag{10}$$

Then as we did in the last post, we can apply the operators to the stationary states, and use the orthogonality of the stationary states to eliminate any integrals involving products of different states. We have

$$\begin{aligned}
\langle x\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\int\Psi^{*}(a_{+}+a_{-})\Psi dx && (11)\\
&= \frac{1}{25}\sqrt{\frac{\hbar}{2m\omega}}\int\left[12\psi_{0}^{2}e^{-i\omega t}+12\psi_{1}^{2}e^{i\omega t}\right]dx && (12)\\
&= \frac{24}{25}\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t) && (13)
\end{aligned}$$

Note that if we replaced ψ_1 with ψ_2 we would get $\langle x\rangle=0$ since all integrals would be over products of different states.

To check Ehrenfest's theorem, we need $\langle p\rangle$ which we can calculate by using the raising and lowering operators again, in the form

$$p=i\sqrt{\frac{\hbar m\omega}{2}}(a_{+}-a_{-}) \tag{14}$$

and we get, doing a similar integral to that for $\langle x\rangle$:

$$\begin{aligned}
\langle p\rangle &= i\sqrt{\frac{\hbar m\omega}{2}}\int\Psi^{*}(a_{+}-a_{-})\Psi dx && (15)\\
&= \frac{i}{25}\sqrt{\frac{\hbar m\omega}{2}}\int\left[12\psi_{1}^{2}e^{i\omega t}-12\psi_{0}^{2}e^{-i\omega t}\right]dx && (16)\\
&= \frac{i}{25}\sqrt{\frac{\hbar m\omega}{2}}(24i\sin\omega t) && (17)\\
&= -\frac{24}{25}\sqrt{\frac{\hbar m\omega}{2}}\sin(\omega t) && (18)
\end{aligned}$$

so

$$\frac{d\langle p\rangle}{dt}=-\frac{24}{25}\sqrt{\frac{\hbar m\omega}{2}}\,\omega\cos(\omega t) \tag{19}$$

To find the mean of the gradient of the potential $V=\frac{1}{2}m\omega^{2}x^{2}$, we can use $\langle x\rangle$:

$$\left\langle \frac{\partial V}{\partial x} \right\rangle = m\omega^2 \langle x \rangle \tag{20}$$

$$= \frac{24}{25}\sqrt{\frac{\hbar m\omega}{2}}\omega\cos(\omega t) \tag{21}$$

$$= -\frac{d\langle p \rangle}{dt} \tag{22}$$

This satisfies Ehrenfest's theorem.

The probabilities of the energies can be read off the original wave function, as they are just the squares of the coefficients of the stationary states. So we will measure E_0 with probability 9/25 and E_1 with probability 16/25.

PINGBACKS

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