HARMONIC OSCILLATOR - PROBABILITY OF BEING OUTSIDE CLASSICAL REGION

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Post date: 19 Jul 2012.

Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.15.

The classical harmonic oscillator has an energy of $E=\frac{1}{2}kx_0^2$ where k is the spring constant and x_0 is the maximum displacement from the equilibrium position. In terms of the frequency of oscillation, this is $E=\frac{1}{2}m\omega^2x_0^2$, so the mass oscillates between $x_0=-\sqrt{2E/m\omega^2}$ and $x_0=\sqrt{2E/m\omega^2}$. For a quantum oscillator, we can work out the probability that the particle is found outside the classical region. In the ground state, we have

$$\psi_0(x)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-m\omega x^2/2\hbar} \tag{1}$$

The probability that the particle is found between two points a and b is

$$P_{ab}=\int_a^b\psi_0^2(x)dx \tag{2}$$

so the probability that the particle is *in* the classical region is

$$P_{classical} = \int_{-\sqrt{2E/m\omega^2}}^{\sqrt{2E/m\omega^2}}\psi_0^2(x)dx \tag{3}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\int_{-\sqrt{2E/m\omega^2}}^{\sqrt{2E/m\omega^2}}e^{-m\omega x^2/\hbar}dx \tag{4}$$

In the ground state, $E=\hbar\omega/2$ so this is

$$P_{classical}=\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\int_{-\sqrt{\hbar/m\omega}}^{\sqrt{\hbar/m\omega}}e^{-m\omega x^2/\hbar}dx \tag{5}$$

This is easier to deal with if we introduce a substitute variable

$$\xi \equiv \sqrt{\frac{m\omega}{\hbar}}x \tag{6}$$

Then the integral transforms to

$$P_{classical} = \frac{1}{\sqrt{\pi}}\int_{-1}^{1} e^{-\xi^2} d\xi \tag{7}$$

This integral is the *error function*, so we get

$$\begin{aligned} P_{classical} &= \mathrm{erf}(1) && (8) \\ &= 0.8427 && (9) \end{aligned}$$

The probability of being outside the classical region is then

$$1 - P_{classical} = 0.1573 \tag{10}$$