

HARMONIC OSCILLATOR - PROBABILITY OF BEING OUTSIDE CLASSICAL REGION

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.15.

The classical harmonic oscillator has an energy of $E = \frac{1}{2}kx_0^2$ where k is the spring constant and x_0 is the maximum displacement from the equilibrium position. In terms of the frequency of oscillation, this is $E = \frac{1}{2}m\omega^2x_0^2$, so the mass oscillates between $x_0 = -\sqrt{2E/m\omega^2}$ and $x_0 = \sqrt{2E/m\omega^2}$. For a quantum oscillator, we can work out the probability that the particle is found outside the classical region. In the ground state, we have

$$(0.1) \quad \psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$$

The probability that the particle is found between two points a and b is

$$(0.2) \quad P_{ab} = \int_a^b \psi_0^2(x) dx$$

so the probability that the particle is *in* the classical region is

$$(0.3) \quad P_{classical} = \int_{-\sqrt{2E/m\omega^2}}^{\sqrt{2E/m\omega^2}} \psi_0^2(x) dx$$

$$(0.4) \quad = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{-\sqrt{2E/m\omega^2}}^{\sqrt{2E/m\omega^2}} e^{-m\omega x^2/\hbar} dx$$

In the ground state, $E = \hbar\omega/2$ so this is

$$(0.5) \quad P_{classical} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{-\sqrt{\hbar/m\omega}}^{\sqrt{\hbar/m\omega}} e^{-m\omega x^2/\hbar} dx$$

This is easier to deal with if we introduce a substitute variable

$$(0.6) \quad \xi \equiv \sqrt{\frac{m\omega}{\hbar}} x$$

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Then the integral transforms to

$$(0.7) \quad P_{classical} = \frac{1}{\sqrt{\pi}} \int_{-1}^1 e^{-\xi^2} d\xi$$

This integral is the *error function*, so we get

$$(0.8) \quad P_{classical} = \operatorname{erf}(1)$$

$$(0.9) \quad = 0.8427$$

The probability of being outside the classical region is then

$$(0.10) \quad 1 - P_{classical} = 0.1573$$