

HARMONIC OSCILLATOR - PROBABILITY OF BEING OUTSIDE CLASSICAL REGION

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Post date: 19 Jul 2012.

Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.15.

The classical harmonic oscillator has an energy of $E = \frac{1}{2}kx_0^2$ where k is the spring constant and x_0 is the maximum displacement from the equilibrium position. In terms of the frequency of oscillation, this is $E = \frac{1}{2}m\omega^2x_0^2$, so the mass oscillates between $x_0 = -\sqrt{2E/m\omega^2}$ and $x_0 = \sqrt{2E/m\omega^2}$. For a quantum oscillator, we can work out the probability that the particle is found outside the classical region. In the ground state, we have

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} \quad (1)$$

The probability that the particle is found between two points a and b is

$$P_{ab} = \int_a^b \psi_0^2(x) dx \quad (2)$$

so the probability that the particle is *in* the classical region is

$$P_{classical} = \int_{-\sqrt{2E/m\omega^2}}^{\sqrt{2E/m\omega^2}} \psi_0^2(x) dx \quad (3)$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{-\sqrt{2E/m\omega^2}}^{\sqrt{2E/m\omega^2}} e^{-m\omega x^2/\hbar} dx \quad (4)$$

In the ground state, $E = \hbar\omega/2$ so this is

$$P_{classical} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{-\sqrt{\hbar/m\omega}}^{\sqrt{\hbar/m\omega}} e^{-m\omega x^2/\hbar} dx \quad (5)$$

This is easier to deal with if we introduce a substitute variable

$$\xi \equiv \sqrt{\frac{m\omega}{\hbar}} x \quad (6)$$

Then the integral transforms to

HARMONIC OSCILLATOR - PROBABILITY OF BEING OUTSIDE CLASSICAL REGION2

$$P_{classical} = \frac{1}{\sqrt{\pi}} \int_{-1}^1 e^{-\xi^2} d\xi \quad (7)$$

This integral is the *error function*, so we get

$$P_{classical} = \text{erf}(1) \quad (8)$$

$$= 0.8427 \quad (9)$$

The probability of being outside the classical region is then

$$1 - P_{classical} = 0.1573 \quad (10)$$