HERMITE POLYNOMIALS - GENERATION

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In the solution of the Schrödinger equation for the harmonic oscillator, we found that the wave function can be expressed as a power series:

$$\psi(y) = e^{-y^2/2} \sum_{j=0}^{\infty} a_j y^j$$

where $y$ was introduced as a shorthand variable:

$$y \equiv \sqrt{\frac{m \omega}{\hbar}} x$$

and the coefficients $a_j$ satisfy the recursion relation

$$a_{j+2} = \frac{2j + 1 - \epsilon}{(j + 1)(j + 2)} a_j$$

where $\epsilon$ is another shorthand variable for the energy:

$$\epsilon \equiv \frac{2E}{\hbar \omega}$$

By requiring the recursion relation to terminate at various values of $j$ we can generate the polynomials for the various energy states, which turn out to be the Hermite polynomials.

For example, if we take the highest value of $j$ to be 5, then we must have

$$a_7 = \frac{11 - \epsilon}{42} a_5$$

$$\epsilon = 11$$

To get the coefficients, we can start with $a_0 = 0$ (since all even terms must be zero if we want $a_5 \neq 0$) and $a_1 = 1$. Then we get
\[ a_{j+2} = \frac{2j - 10}{(j + 1)(j + 2)} a_j \]  

so \( a_3 = -\frac{4}{3} \), \( a_5 = \left( -\frac{1}{3} \right) \left( -\frac{4}{3} \right) = \frac{4}{15} \). The Hermite polynomial is

\[ H_5(x) = A_5 \left[ \frac{4}{15} x^5 - \frac{4}{3} x^3 + x \right] \]

where \( A_5 \) is a constant that is set by convention to make the coefficients satisfy some specified rule.

If we require the coefficient of the highest power \( n \) to be \( 2^n \) we can multiply this polynomial by 120 to get

\[ H_5(x) = 32x^5 - 160x^3 + 120x \]

For \( H_6 \), all the odd terms are zero, and we require \( a_8 = 0 \), so we get

\[ \epsilon = 13 \]
\[ E = \frac{13}{2} \hbar \omega \]
\[ a_{j+2} = \frac{2j - 12}{(j + 1)(j + 2)} a_j \]

Taking \( a_0 = 1 \) and \( a_1 = 0 \), we get \( a_2 = -6 \), \( a_4 = 4 \), \( a_6 = -\frac{8}{15} \). Requiring the coefficient of \( a_6 \) to be \( 2^6 = 64 \) we get

\[ H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120 \]