

## HERMITE POLYNOMIALS - GENERATION

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.16.

In the solution of the Schrödinger equation for the harmonic oscillator, we found that the wave function can be expressed as a power series:

$$(1) \quad \psi(y) = e^{-y^2/2} \sum_{j=0}^{\infty} a_j y^j$$

where  $y$  was introduced as a shorthand variable:

$$(2) \quad y \equiv \sqrt{\frac{m\omega}{\hbar}} x$$

and the coefficients  $a_j$  satisfy the recursion relation

$$(3) \quad a_{j+2} = \frac{2j+1-\varepsilon}{(j+1)(j+2)} a_j$$

where  $\varepsilon$  is another shorthand variable for the energy:

$$(4) \quad \varepsilon \equiv \frac{2E}{\hbar\omega}$$

By requiring the recursion relation to terminate at various values of  $j$  we can generate the polynomials for the various energy states, which turn out to be the Hermite polynomials.

For example, if we take the highest value of  $j$  to be 5, then we must have

$$(5) \quad a_7 = \frac{11-\varepsilon}{42} a_5$$

$$(6) \quad = 0$$

$$(7) \quad \varepsilon = 11$$

$$(8) \quad E = \frac{11}{2} \hbar\omega$$

To get the coefficients, we can start with  $a_0 = 0$  (since all even terms must be zero if we want  $a_5 \neq 0$ ) and  $a_1 = 1$ . Then we get

$$(9) \quad a_{j+2} = \frac{2j-10}{(j+1)(j+2)} a_j$$

so  $a_3 = -\frac{4}{3}$ ,  $a_5 = \left(-\frac{1}{5}\right) \left(-\frac{4}{3}\right) = \frac{4}{15}$ . The Hermite polynomial is

$$(10) \quad H_5(x) = A_5 \left[ \frac{4}{15}x^5 - \frac{4}{3}x^3 + x \right]$$

where  $A_5$  is a constant that is set by convention to make the coefficients satisfy some specified rule.

If we require the coefficient of the highest power  $n$  to be  $2^n$  we can multiply this polynomial by 120 to get

$$(11) \quad H_5(x) = 32x^5 - 160x^3 + 120x$$

For  $H_6$ , all the odd terms are zero, and we require  $a_8 = 0$ , so we get

$$(12) \quad \varepsilon = 13$$

$$(13) \quad E = \frac{13}{2} \hbar \omega$$

$$(14) \quad a_{j+2} = \frac{2j-12}{(j+1)(j+2)} a_j$$

Taking  $a_0 = 1$  and  $a_1 = 0$ , we get  $a_2 = -6$ ,  $a_4 = 4$ ,  $a_6 = -\frac{8}{15}$ . Requiring the coefficient of  $a_6$  to be  $2^6 = 64$  we get

$$(15) \quad H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120$$

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