HERMITE POLYNOMIALS - GENERATION

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.16.

In the solution of the Schrödinger equation for the harmonic oscillator, we found that the wave function can be expressed as a power series:

$$\psi(y)=e^{-y^{2}/2}\sum_{j=0}^{\infty}a_{j}y^{j} \tag{0.1}$$

where y was introduced as a shorthand variable:

$$y\equiv\sqrt{\frac{m\omega}{\hbar}}x \tag{0.2}$$

and the coefficients a_j satisfy the recursion relation

$$a_{j+2}=\frac{2j+1-\varepsilon}{(j+1)(j+2)}a_{j} \tag{0.3}$$

where ε is another shorthand variable for the energy:

$$\varepsilon\equiv\frac{2E}{\hbar\omega} \tag{0.4}$$

By requiring the recursion relation to terminate at various values of j we can generate the polynomials for the various energy states, which turn out to be the Hermite polynomials.

For example, if we take the highest value of j to be 5, then we must have

$$a_{7} = \frac{11-\varepsilon}{42}a_{5} \tag{0.5}$$
$$= 0 \tag{0.6}$$
$$\varepsilon = 11 \tag{0.7}$$
$$E = \frac{11}{2}\hbar\omega \tag{0.8}$$

To get the coefficients, we can start with $a_0 = 0$ (since all even terms must be zero if we want $a_5 \neq 0$) and $a_1 = 1$. Then we get

$$a_{j+2} = \frac{2j-10}{(j+1)(j+2)} a_j \tag{0.9}$$

so $a_3 = -\frac{4}{3}$, $a_5 = \left(-\frac{1}{5}\right)\left(-\frac{4}{3}\right) = \frac{4}{15}$. The Hermite polynomial is

$$H_5(x) = A_5 \left[\frac{4}{15}x^5 - \frac{4}{3}x^3 + x\right] \tag{0.10}$$

where A_5 is a constant that is set by convention to make the coefficients satisfy some specified rule.

If we require the coefficient of the highest power n to be 2^n we can multiply this polynomial by 120 to get

$$H_5(x) = 32x^5 - 160x^3 + 120x \tag{0.11}$$

For H_6, all the odd terms are zero, and we require $a_8 = 0$, so we get

$$\varepsilon = 13 \tag{0.12}$$

$$E = \frac{13}{2}\hbar\omega \tag{0.13}$$

$$a_{j+2} = \frac{2j-12}{(j+1)(j+2)} a_j \tag{0.14}$$

Taking $a_0 = 1$ and $a_1 = 0$, we get $a_2 = -6$, $a_4 = 4$, $a_6 = -\frac{8}{15}$. Requiring the coefficient of a_6 to be $2^6 = 64$ we get

$$H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120 \tag{0.15}$$