HERMITE POLYNOMIALS - RECURSION RELATIONS

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.17(b-d).

Here are some more theorems concerning Hermite polynomials, which show up in the solution of the Schrödinger equation for the harmonic oscillator.

The first theorem is that the Hermite polynomials can be obtained from a *generating function*. We've seen generating functions in the context of the Laguerre polynomials, which occur in the physics of the hydrogen atom. The derivation of generating functions is something of a black art, and as it requires the use of complex variable theory (in particular, Cauchy's integral formula) we'll just accept it without proof for now. The result is

$$e^{-z^2+2zx}=\sum_{n=0}^{\infty}\frac{z^n}{n!}H_n(x) \tag{1}$$

Here z is a dummy variable which is used to generate the Taylor series of the exponential on the left. Since the kth derivative with respect to z of the series eliminates all powers with $n<k$, retains z^{n-k} terms for $n>k$ and reduces the term in z^k to $(k!/k!)H_k(x)=H_k(x)$, if we take the kth derivative and then set $z=0$ we're left, magically, with $H_k(x)$. Taking high order derivatives of the exponential isn't exactly pretty, of course, but it's quite marvellous that such a result exists at all.

As an example, we'll use the generating function to derive the first three polynomials. We get

$$\begin{aligned}
H_0(x) &= e^0 && (2)\\
&= 1 && (3)\\
H_1(x) &= \left.\frac{d}{dz}e^{-z^2+2zx}\right|_{z=0} && (4)\\
&= \left.(-2z+2x)\,e^{-z^2+2zx}\right|_{z=0} && (5)\\
&= 2x && (6)\\
H_2(x) &= \left.\frac{d^2}{dz^2}e^{-z^2+2zx}\right|_{z=0} && (7)\\
&= \left.\left[-2+(-2z+2x)^2\right]e^{-z^2+2zx}\right|_{z=0} && (8)\\
&= 4x^2-2 && (9)
\end{aligned}$$

Starting from the generating function, we can derive two recursion relations for the polynomials. If we take the derivative of 1 with respect to z we get

$$(-2z+2x)\,e^{-z^2+2zx} = \sum_{n=1}^{\infty}\frac{z^{n-1}}{(n-1)!}H_n(x) \tag{10}$$

If we replace the exponential on the left by its series expansion, we get

$$-2\sum_{n=0}^{\infty}\frac{z^{n+1}}{n!}H_n(x)+2x\sum_{n=0}^{\infty}\frac{z^n}{n!}H_n(x) = \sum_{n=1}^{\infty}\frac{z^{n-1}}{(n-1)!}H_n(x) \tag{11}$$

We now pull the usual trick of relabelling the summation index on the first and last sum in order to make the power of z the same in all sums. For the first sum, we get

$$\begin{aligned}
-2\sum_{n=0}^{\infty}\frac{z^{n+1}}{n!}H_n(x) &= -2\sum_{n=1}^{\infty}\frac{z^n}{(n-1)!}H_{n-1}(x) && (12)\\
&= -2\sum_{n=1}^{\infty}n\frac{z^n}{n!}H_{n-1}(x) && (13)
\end{aligned}$$

For the sum on the right, we get

$$\sum_{n=1}^{\infty}\frac{z^{n-1}}{(n-1)!}H_n(x) = \sum_{n=0}^{\infty}\frac{z^n}{n!}H_{n+1}(x) \tag{14}$$

Combining these results we get

$$-2\sum_{n=1}^{\infty}n\frac{z^n}{n!}H_{n-1}(x)+2x\sum_{n=0}^{\infty}\frac{z^n}{n!}H_n(x)=\sum_{n=0}^{\infty}\frac{z^n}{n!}H_{n+1}(x) \tag{15}$$

For $n>0$, we can equate the coefficients of z^n to get

$$H_{n+1}(x)=2xH_n(x)-2nH_{n-1}(x) \tag{16}$$

For the special case of $n=0$ the first sum on the left makes no contribution and the other two terms give us

$$2xH_0(x)=H_1(x) \tag{17}$$

which, since $H_0=1$, gives us $H_1=2x$, which is correct. We saw when discussing the Rodrigues formula that

$$H_3(x) = 8x^3-12x \tag{18}$$
$$H_4(x) = 16x^4-48x^2+12 \tag{19}$$

so we can use the recursion relation to get the next couple of polynomials:

$$H_5(x) = 2xH_4(x)-8H_3(x) \tag{20}$$
$$= 32x^5-96x^3+24x-64x^3+96x \tag{21}$$
$$= 32x^5-160x^3+120x \tag{22}$$
$$H_6(x) = 2xH_5(x)-10H_4(x) \tag{23}$$
$$= 64x^6-320x^4+240x^2-160x^4+480x^2-120 \tag{24}$$
$$= 64x^6-480x^4+720x^2-120 \tag{25}$$

A second recursion relation can be found by differentiating 1 with respect to x rather than z. We get

$$2ze^{-z^2+2zx}=\sum_{n=0}^{\infty}H_n'(x)\frac{z^n}{n!} \tag{26}$$

Again, we replace the exponential by the series to get

$$2\sum_{n=0}^{\infty}H_n(x)\frac{z^{n+1}}{n!}=\sum_{n=0}^{\infty}H_n'(x)\frac{z^n}{n!} \tag{27}$$

Relabelling the summation index on the left, we get

$$2\sum_{n=1}^{\infty}H_{n-1}(x)\frac{z^{n}}{(n-1)!}=\sum_{n=0}^{\infty}H_{n}^{\prime}(x)\frac{z^{n}}{n!} \tag{28}$$

Equating coefficients of z^n we get

$$H_{n}^{\prime}(x)=2nH_{n-1}(x) \tag{29}$$

For example:

$$H_{6}^{\prime} = 384x^{5}-1920x^{3}+1440x \tag{30}$$
$$= 12\left(32x^{5}-160x^{3}+120x\right) \tag{31}$$
$$= 2\times6H_{5} \tag{32}$$
$$H_{5}^{\prime} = 160x^{4}-480x^{2}+120 \tag{33}$$
$$= 10\left(16x^{4}-48x^{2}+12\right) \tag{34}$$
$$= 2\times5H_{4} \tag{35}$$

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