PLANCHEREL'S THEOREM

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.20.

While analyzing the free particle, we saw that we could construct a normalizable combination of stationary states by writing

$$\Psi(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\phi(k)e^{ikx}e^{-i\hbar k^2t/2m}dk \tag{1}$$

We can find the function $\phi(k)$ by specifying the initial wave function:

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\phi(k)e^{ikx}dk \tag{2}$$

This relation can be inverted by using Plancherel's theorem, which states

$$\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\Psi(x,0)e^{-ikx}dx \tag{3}$$

Here we run through a plausibility argument which is a sort of physicist's proof of Plancherel's theorem. We start with Dirichlet's theorem which says that any (physically realistic, anyway) function can be written as a Fourier series. We can show that this is equivalent to a series in complex exponentials. That is

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty}c_n e^{in\pi x/a} &(4)\\ &= \sum_{n=-\infty}^{\infty}c_n\left[\cos\frac{n\pi x}{a}+i\sin\frac{n\pi x}{a}\right] &(5)\\ &= c_0+\sum_{n=1}^{\infty}(c_n+c_{-n})\cos\frac{n\pi x}{a}+i\sum_{n=1}^{\infty}(c_n-c_{-n})\sin\frac{n\pi x}{a} &(6)\end{aligned}$$

We've used the facts that cosine is even and sine is odd. This is equivalent to a Fourier series:

$$f(x)=\sum_{n=0}^{\infty}\left[a_n\sin\frac{n\pi x}{a}+b_n\cos\frac{n\pi x}{a}\right] \tag{7}$$

where the coefficients are related by

$$\begin{aligned} b_0 &= c_0 && (8)\\ b_n &= c_n+c_{-n} && (9)\\ a_n &= i\left(c_n-c_{-n}\right) && (10)\end{aligned}$$

Inverting the relations we get, for $n>0$

$$\begin{aligned} c_n &= \frac{1}{2}\left(b_n-ia_n\right) && (11)\\ c_{-n} &= \frac{1}{2}\left(b_n+ia_n\right) && (12)\end{aligned}$$

We can get the coefficients in terms of $f(x)$ by integration:

$$\frac{1}{2a}\int_{-a}^{a}f(x)e^{-in\pi x/a}dx=\frac{1}{2a}\sum_{m=-\infty}^{\infty}c_m\int_{-a}^{a}e^{i(m-n)\pi x/a}dx \tag{13}$$

The integral is zero if $m\neq n$ and $2a$ if $m=n$, so the right hand side comes out to just c_n and we get

$$c_n=\frac{1}{2a}\int_{-a}^{a}f(x)e^{-in\pi x/a}dx \tag{14}$$

Now we can make the substitutions

$$\begin{aligned} k &\equiv \frac{n\pi}{a} && (15)\\ F(k) &\equiv \sqrt{\frac{2}{\pi}}ac_n && (16)\end{aligned}$$

If Δk is the increment in k from one n to the next, then $\Delta k=\pi/a$. We can then write the original series as

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty}\sqrt{\frac{\pi}{2}}\frac{1}{a}F(k)e^{ikx}\left(\frac{a}{\pi}\Delta k\right) && (17)\\ &= \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}F(k)e^{ikx}\Delta k && (18)\end{aligned}$$

The formula for c_n now becomes

$$\sqrt{\frac{\pi}{2}}\frac{1}{a}F(k) = \frac{1}{2a}\int_{-a}^{a}f(x)e^{-ikx}dx \tag{19}$$

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}f(x)e^{-ikx}dx \tag{20}$$

Now we can take the limit as $a \to \infty$. In this case, $\Delta k \to dk$ (that is, it becomes a differential) and the sum becomes an integral, so we get

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(k)e^{ikx}dk \tag{21}$$

In the second formula, the limits on the integral become infinite, and we get the other half of Plancherel's theorem:

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-ikx}dx \tag{22}$$

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