

PLANCHEREL'S THEOREM

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.20.

While analyzing the free particle, we saw that we could construct a normalizable combination of stationary states by writing

$$(0.1) \quad \Psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k) e^{ikx} e^{-i\hbar k^2 t / 2m} dk$$

We can find the function $\phi(k)$ by specifying the initial wave function:

$$(0.2) \quad \Psi(x, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k) e^{ikx} dk$$

This relation can be inverted by using Plancherel's theorem, which states

$$(0.3) \quad \phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(x, 0) e^{-ikx} dx$$

Here we run through a plausibility argument which is a sort of physicist's proof of Plancherel's theorem. We start with Dirichlet's theorem which says that any (physically realistic, anyway) function can be written as a Fourier series. We can show that this is equivalent to a series in complex exponentials. That is

$$(0.4) \quad f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/a}$$

$$(0.5) \quad = \sum_{n=-\infty}^{\infty} c_n \left[\cos \frac{n\pi x}{a} + i \sin \frac{n\pi x}{a} \right]$$

$$(0.6) \quad = c_0 + \sum_{n=1}^{\infty} (c_n + c_{-n}) \cos \frac{n\pi x}{a} + i \sum_{n=1}^{\infty} (c_n - c_{-n}) \sin \frac{n\pi x}{a}$$

We've used the facts that cosine is even and sine is odd. This is equivalent to a Fourier series:

$$(0.7) \quad f(x) = \sum_{n=0}^{\infty} \left[a_n \sin \frac{n\pi x}{a} + b_n \cos \frac{n\pi x}{a} \right]$$

where the coefficients are related by

$$(0.8) \quad b_0 = c_0$$

$$(0.9) \quad b_n = c_n + c_{-n}$$

$$(0.10) \quad a_n = i(c_n - c_{-n})$$

Inverting the relations we get, for $n > 0$

$$(0.11) \quad c_n = \frac{1}{2}(b_n - ia_n)$$

$$(0.12) \quad c_{-n} = \frac{1}{2}(b_n + ia_n)$$

We can get the coefficients in terms of $f(x)$ by integration:

$$(0.13) \quad \frac{1}{2a} \int_{-a}^a f(x) e^{-in\pi x/a} dx = \frac{1}{2a} \sum_{m=-\infty}^{\infty} c_m \int_{-a}^a e^{i(m-n)\pi x/a} dx$$

The integral is zero if $m \neq n$ and $2a$ if $m = n$, so the right hand side comes out to just c_n and we get

$$(0.14) \quad c_n = \frac{1}{2a} \int_{-a}^a f(x) e^{-in\pi x/a} dx$$

Now we can make the substitutions

$$(0.15) \quad k \equiv \frac{n\pi}{a}$$

$$(0.16) \quad F(k) \equiv \sqrt{\frac{2}{\pi}} a c_n$$

If Δk is the increment in k from one n to the next, then $\Delta k = \pi/a$. We can then write the original series as

$$(0.17) \quad f(x) = \sum_{n=-\infty}^{\infty} \sqrt{\frac{\pi}{2}} \frac{1}{a} F(k) e^{ikx} \left(\frac{a}{\pi} \Delta k \right)$$

$$(0.18) \quad = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} F(k) e^{ikx} \Delta k$$

The formula for c_n now becomes

$$(0.19) \quad \sqrt{\frac{\pi}{2}} \frac{1}{a} F(k) = \frac{1}{2a} \int_{-a}^a f(x) e^{-ikx} dx$$

$$(0.20) \quad F(k) = \frac{1}{\sqrt{2\pi}} \int_{-a}^a f(x) e^{-ikx} dx$$

Now we can take the limit as $a \rightarrow \infty$. In this case, $\Delta k \rightarrow dk$ (that is, it becomes a differential) and the sum becomes an integral, so we get

$$(0.21) \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx} dk$$

In the second formula, the limits on the integral become infinite, and we get the other half of Plancherel's theorem:

$$(0.22) \quad F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

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