

DIRAC DELTA FUNCTION

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.24.

One of the weirder bits of mathematics that the physics student will encounter is the Dirac delta function $\delta(x)$. In one dimension, the 'function' (technically it's really not a function at all, but a distribution) can be defined by saying $\delta(x) = 0$ for all $x \neq 0$ but $\delta(x) = \infty$ at $x = 0$. However, this definition isn't very satisfactory, and in fact doesn't define $\delta(x)$ uniquely. It is better to define it using two conditions:

$$(1) \quad \delta(x) = 0 \text{ if } x \neq 0$$
$$(2) \quad \int_{-\infty}^{\infty} \delta(x) dx = 1$$

It is the integral condition which pins the delta function down uniquely.

For those that like to visualize functions, the delta function can be thought of as a limit of a series of rectangular functions. If we define a rectangular function of width $1/w$ and height w (so its area is 1) centred horizontally on the y -axis and sitting with its base on the x -axis, then we can visualize the delta function as the limit of this rectangle as $w \rightarrow \infty$. As w gets larger, the rectangle gets taller and thinner, and in the limit it is an infinitely high spike with only an infinitesimal width, but with an area always equal to 1.

One consequence of the delta function being zero everywhere except at $x = 0$ is that if we multiply it by any function, it doesn't matter what that function's values are except at $x = 0$. That is, we can say

$$(3) \quad f(x)\delta(x) = f(0)\delta(x)$$

In terms of integrals, this means that

$$\begin{aligned}
 (4) \quad \int_{-\infty}^{\infty} f(x)\delta(x)dx &= \int_{-\infty}^{\infty} f(0)\delta(x)dx \\
 (5) &= f(0) \int_{-\infty}^{\infty} \delta(x)dx \\
 (6) &= f(0)
 \end{aligned}$$

using the second defining property of $\delta(x)$ above. The effect of integrating a function multiplied by the delta function is to pick out the function's value at $x = 0$.

It is easy enough to move the location of the delta function's spike. If we want the spike to appear at $x = a$ we can use the function $\delta(x - a)$, since the spike occurs when the delta function's argument is zero, that is, at $x - a = 0$. Thus we can generalize the integral formula above to

$$(7) \quad \int_{-\infty}^{\infty} f(x)\delta(x - a)dx = f(a)$$

One of the trickier formulas that causes some consternation amongst physics students is this formula:

$$(8) \quad \delta(kx) = \frac{1}{|k|}\delta(x)$$

where k is a non-zero constant.

Before we discuss what this means, we can run through the proof. Since the main use of delta functions is in integration, we can consider this formula in light of the integration condition. Suppose we define the variable transformation $y = kx$. Then $dy = k dx$. We'll take $k > 0$ first:

$$\begin{aligned}
 (9) \quad \int_{-\infty}^{\infty} f(x)\delta(kx)dx &= \int_{-\infty}^{\infty} f(y/k)\delta(y)\frac{1}{k}dy \\
 (10) &= \frac{1}{k} \int_{-\infty}^{\infty} f(y/k)\delta(y)dy \\
 (11) &= \frac{1}{k}f(0)
 \end{aligned}$$

This is the same result as what we would get if we evaluated:

$$(12) \quad \frac{1}{k} \int_{-\infty}^{\infty} f(x)\delta(x)dx = \frac{1}{k}f(0)$$

so it seems reasonable to take $\delta(kx) = \delta(x)/k$ in this case.

If $k < 0$, then the derivation is the same except that making the variable substitution $y = kx$ inverts the limits of integration, so we get

$$(13) \quad \int_{-\infty}^{\infty} f(x)\delta(kx)dx = \int_{\infty}^{-\infty} f(y/k)\delta(y)\frac{1}{k}dy$$

$$(14) \quad = -\int_{-\infty}^{\infty} f(y/k)\delta(y)\frac{1}{k}dy$$

$$(15) \quad = -\frac{1}{k}f(0)$$

$$(16) \quad = \frac{1}{|k|}f(0)$$

Therefore, saying $\delta(kx) = \delta(x)/|k|$ covers both cases.

Now the problem many people have with this formula is this: if the definition of $\delta(x)$ is that it is zero everywhere, but infinite when $x = 0$, then since the same can be said of $\delta(kx)$, why can't we just say $\delta(kx) = \delta(x)$?

The reason arises in the ambiguity of this non-integral definition of the delta function that I mentioned at the start. There are many ways we can define a function that is zero everywhere but infinite at $x = 0$. Look at it this way. Instead of using the limit of the sequence of rectangles that I did at the start, suppose we use a sequence of rectangles of width $1/kw$ and height w , so that their areas are all $1/k$. Now in the limit as $w \rightarrow \infty$ we get another function that is zero everywhere except at $x = 0$ and has an infinite spike at $x = 0$ but it is clearly not the same as $\delta(x)$ since the area of the spike is $1/k$ instead of 1. This new function which results from scaling the x axis using the transformation $x \rightarrow kx$ gives you a spike with an area of $1/k$ times the original.

A generalization of this formula is

$$(17) \quad \delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|} \delta(x)$$

The proof of this follows the same lines as above. We consider the case $f'(x_0) > 0$ and do the integral

$$(18) \quad \int_{-\infty}^{\infty} g(x)\delta(f(x) - f(x_0))dx$$

Now do the substitution

$$(19) \quad y = f(x) - f(x_0)$$

$$(20) \quad dy = f'(x) dx$$

$$(21) \quad x = f^{-1}(y + f(x_0))$$

Then we get

$$(22) \quad \int_{-\infty}^{\infty} g(x) \delta(f(x) - f(x_0)) dx = \int_{-\infty}^{\infty} g(f^{-1}(y + f(x_0))) \frac{\delta(y)}{f'(x)} dy$$

Since $y = 0$ when $x = x_0$ we get

$$(23) \quad \int_{-\infty}^{\infty} g(f^{-1}(y + f(x_0))) \frac{\delta(y)}{f'(x)} dy = g(f^{-1}(f(x_0))) \frac{1}{f'(x_0)}$$

$$(24) \quad = \frac{g(x_0)}{f'(x_0)}$$

As above when $k < 0$, if $f'(x_0) < 0$ we get $-g(x_0)/f'(x_0)$ so the general formula is as given in 17.

Another formula that can cause nightmares is the derivative of the step function, that is of the function

$$(25) \quad H(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

Since the function is constant everywhere except at $x = 0$ its derivative is zero everywhere except at $x = 0$. However, the step function is discontinuous at this point, and since it jumps a finite amount over a single point, it would seem that its derivative is infinite at that point. To see what's going on, suppose we try the integral

$$(26) \quad \int_{-1}^1 f(x) \frac{dH}{dx} dx = f(0) \int_{-1}^1 \frac{dH}{dx} dx$$

$$(27) \quad = f(0)[H(1) - H(-1)]$$

$$(28) \quad = f(0)$$

(the limits on the integral could be any interval which includes 0). The first line uses the fact that $dH/dx = 0$ everywhere except $x = 0$, and the second line is just the ordinary evaluation of an integral.

Thus dH/dx satisfies both conditions of the delta function, so we can say

$$(29) \quad \frac{dH}{dx} = \delta(x)$$

Note that if the step is a different size, say $k > 0$, so that we have

$$(30) \quad H_k(x) = \begin{cases} 0 & x \leq 0 \\ k & x > 0 \end{cases}$$

then the same analysis gives

$$(31) \quad \int_{-1}^1 f(x) \frac{dH_k}{dx} dx = kf(0)$$

so from the earlier example, we get

$$(32) \quad \frac{dH_k}{dx} = \delta(x/k)$$

$$(33) \quad = k\delta(x)$$