

## DELTA FUNCTION WELL: BOUND STATE - UNCERTAINTY PRINCIPLE

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.25.

We've seen that the bound state of a particle in a delta function potential well has the wave function

$$(1) \quad \psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2}$$

We can work out the mean values of position and momentum, and of their squares, in the usual way.

For  $\langle x \rangle$ , since the wave function is even and the function  $f(x) = x$  is odd, we have

$$(2) \quad \langle x \rangle = \int_{-\infty}^{\infty} x \psi^2(x) dx$$

$$(3) \quad = 0$$

since the integrand is odd.

For the mean square position, we get

$$(4) \quad \langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \psi^2(x) dx$$

$$(5) \quad = \frac{m\alpha}{\hbar^2} \int_{-\infty}^{\infty} x^2 e^{-2m\alpha|x|/\hbar^2} dx$$

$$(6) \quad = \frac{2m\alpha}{\hbar^2} \int_0^{\infty} x^2 e^{-2m\alpha x/\hbar^2} dx$$

$$(7) \quad = \frac{\hbar^4}{2m^2\alpha^2}$$

where we used software to do the integral.

Now for the momentum. We get

$$(8) \quad \langle p \rangle = \frac{\hbar m\alpha}{i \hbar^2} \int_{-\infty}^{\infty} e^{-m\alpha|x|/\hbar^2} \frac{d}{dx} e^{-m\alpha|x|/\hbar^2} dx$$

We can split this integral into two parts joined at the origin.

$$(9) \quad \frac{d}{dx} e^{-m\alpha|x|/\hbar^2} = \begin{cases} -\frac{m\alpha}{\hbar^2} e^{-m\alpha x/\hbar^2} & x > 0 \\ \frac{m\alpha}{\hbar^2} e^{m\alpha x/\hbar^2} & x < 0 \end{cases}$$

The derivative is therefore an odd function. Since the original wave function is even, the product of the two is odd, so  $\langle p \rangle = 0$ .

Calculating  $\langle p^2 \rangle$  is a bit trickier, since the derivative above is discontinuous at the origin. If we define the step function

$$(10) \quad H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

we can write the derivative above as

$$(11) \quad \frac{d}{dx} e^{-m\alpha|x|/\hbar^2} = -\frac{m\alpha}{\hbar^2} e^{-m\alpha|x|/\hbar^2} (2H(x) - 1)$$

We've seen that the derivative of the step function can be taken as the delta function

$$(12) \quad \frac{dH}{dx} = \delta(x)$$

so the second derivative of the wave function is

$$(13) \quad \frac{d^2}{dx^2} \psi(x) = \frac{\sqrt{m\alpha}}{\hbar} \frac{d^2}{dx^2} e^{-m\alpha|x|/\hbar^2}$$

$$(14) \quad = -\frac{(m\alpha)^{3/2}}{\hbar^3} \frac{d}{dx} \left[ e^{-m\alpha|x|/\hbar^2} (2H(x) - 1) \right]$$

$$(15) \quad = \frac{(m\alpha)^{5/2}}{\hbar^5} e^{-m\alpha|x|/\hbar^2} (2H(x) - 1)^2 - 2 \frac{(m\alpha)^{3/2}}{\hbar^3} e^{-m\alpha|x|/\hbar^2} \delta(x)$$

Observing that  $(2H(x) - 1)^2 = 1$  everywhere, we get

$$(16) \quad \frac{d^2}{dx^2} \psi(x) = e^{-m\alpha|x|/\hbar^2} \left[ \frac{(m\alpha)^{5/2}}{\hbar^5} - 2 \frac{(m\alpha)^{3/2}}{\hbar^3} \delta(x) \right]$$

After all this we can now calculate  $\langle p^2 \rangle$ :

$$(17) \quad \langle p^2 \rangle = \left( \frac{\hbar}{i} \right)^2 \int_{-\infty}^{\infty} \psi(x) \frac{d^2}{dx^2} \psi(x) dx$$

$$(18) \quad = -\hbar^2 \frac{\sqrt{m\alpha}}{\hbar} \int_{-\infty}^{\infty} e^{-2m\alpha|x|/\hbar^2} \left[ \frac{(m\alpha)^{5/2}}{\hbar^5} - 2 \frac{(m\alpha)^{3/2}}{\hbar^3} \delta(x) \right] dx$$

$$(19) \quad = -\frac{(m\alpha)^3}{\hbar^4} \int_{-\infty}^{\infty} e^{-2m\alpha|x|/\hbar^2} dx + 2 \left( \frac{m\alpha}{\hbar} \right)^2$$

$$(20) \quad = -2 \frac{(m\alpha)^3}{\hbar^4} \int_0^{\infty} e^{-2m\alpha x/\hbar^2} dx + 2 \left( \frac{m\alpha}{\hbar} \right)^2$$

$$(21) \quad = \left( \frac{m\alpha}{\hbar} \right)^2$$

Going from the second to the third line, we used the property of the delta function

$$(22) \quad \int_{-\infty}^{\infty} f(x) \delta(x) dx = f(0)$$

The uncertainty principle for the bound state of the delta function potential is therefore

$$(23) \quad \sigma_x \sigma_p = \sqrt{\langle x^2 \rangle \langle p^2 \rangle}$$

$$(24) \quad = \frac{\hbar}{\sqrt{2}}$$

COMMENTS

*Remark 1.* From h1adacpravdy; Nov 4, 2017 2:45 PM

Equation 21: = (m \* alpha / hb)^2

From this follows: sigma\_p = sqrt(-^2) = m \* alpha / hb

Unit analysis: m ... kg alpha ... J hb ... Js m \* alpha / hb ..... kg/s  
SI unit for momentum is kg m/s. Where I have done mistake ?

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$\alpha$  has units of Energy  $\times$  Length - the exponent in 1 must be dimensionless.