DELTA FUNCTION WELL: BOUND STATE - UNCERTAINTY PRINCIPLE

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.25.

We've seen that the bound state of a particle in a delta function potential wellhas the wave function

$$\psi(x)=\frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha|x|/\hbar^{2}} \tag{0.1}$$

We can work out the mean values of position and momentum, and of their squares, in the usual way.

For $\langle x\rangle$, since the wave function is even and the function $f(x)=x$ is odd, we have

$$\langle x\rangle = \int_{-\infty}^{\infty}x\psi^{2}(x)dx \tag{0.2}$$

$$= 0 \tag{0.3}$$

since the integrand is odd.

For the mean square position, we get

$$\left\langle x^{2}\right\rangle = \int_{-\infty}^{\infty}x^{2}\psi^{2}(x)dx \tag{0.4}$$

$$= \frac{m\alpha}{\hbar^{2}}\int_{-\infty}^{\infty}x^{2}e^{-2m\alpha|x|/\hbar^{2}}dx \tag{0.5}$$

$$= \frac{2m\alpha}{\hbar^{2}}\int_{0}^{\infty}x^{2}e^{-2m\alpha x/\hbar^{2}}dx \tag{0.6}$$

$$= \frac{\hbar^{4}}{2m^{2}\alpha^{2}} \tag{0.7}$$

where we used software to do the integral.

Now for the momentum. We get

$$\langle p\rangle = \frac{\hbar}{i}\frac{m\alpha}{\hbar^2}\int_{-\infty}^{\infty} e^{-m\alpha|x|/\hbar^2}\frac{d}{dx}e^{-m\alpha|x|/\hbar^2}dx \tag{0.8}$$

We can split this integral into two parts joined at the origin.

$$\frac{d}{dx}e^{-m\alpha|x|/\hbar^2} = \begin{cases} -\frac{m\alpha}{\hbar^2}e^{-m\alpha x/\hbar^2} & x>0 \\ \frac{m\alpha}{\hbar^2}e^{m\alpha x/\hbar^2} & x<0 \end{cases} \tag{0.9}$$

The derivative is therefore an odd function. Since the original wave function is even, the product of the two is odd, so $\langle p\rangle = 0$.

Calculating $\left\langle p^2\right\rangle$ is a bit trickier, since the derivative above is discontinuous at the origin. If we define the step function

$$H(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases} \tag{0.10}$$

we can write the derivative above as

$$\frac{d}{dx}e^{-m\alpha|x|/\hbar^2} = -\frac{m\alpha}{\hbar^2}e^{-m\alpha|x|/\hbar^2}\left(2H(x)-1\right) \tag{0.11}$$

We've seen that the derivative of the step functioncan be taken as the delta function

$$\frac{dH}{dx} = \delta(x) \tag{0.12}$$

so the second derivative of the wave function is

$$\frac{d^2}{dx^2}\psi(x) = \frac{\sqrt{m\alpha}}{\hbar}\frac{d^2}{dx^2}e^{-m\alpha|x|/\hbar^2} \tag{0.13}$$

$$= -\frac{(m\alpha)^{3/2}}{\hbar^3}\frac{d}{dx}\left[e^{-m\alpha|x|/\hbar^2}\left(2H(x)-1\right)\right] \tag{0.14}$$

$$= \frac{(m\alpha)^{5/2}}{\hbar^5}e^{-m\alpha|x|/\hbar^2}\left(2H(x)-1\right)^2 - 2\frac{(m\alpha)^{3/2}}{\hbar^3}e^{-m\alpha|x|/\hbar^2}\delta(x) \tag{0.15}$$

Observing that $(2H(x)-1)^2 = 1$ everywhere, we get

$$\frac{d^2}{dx^2}\psi(x) = e^{-m\alpha|x|/\hbar^2}\left[\frac{(m\alpha)^{5/2}}{\hbar^5} - 2\frac{(m\alpha)^{3/2}}{\hbar^3}\delta(x)\right] \tag{0.16}$$

After all this we can now calculate $\left\langle p^2\right\rangle$:

$$\left\langle p^2\right\rangle = \left(\frac{\hbar}{i}\right)^2\int_{-\infty}^{\infty}\psi(x)\frac{d^2}{dx^2}\psi(x)dx \tag{0.17}$$

$$= -\hbar^2\frac{\sqrt{m\alpha}}{\hbar}\int_{-\infty}^{\infty}e^{-2m\alpha|x|/\hbar^2}\left[\frac{(m\alpha)^{5/2}}{\hbar^5} - 2\frac{(m\alpha)^{3/2}}{\hbar^3}\delta(x)\right]dx \tag{0.18}$$

$$= -\frac{(m\alpha)^3}{\hbar^4}\int_{-\infty}^{\infty}e^{-2m\alpha|x|/\hbar^2}dx + 2\left(\frac{m\alpha}{\hbar}\right)^2 \tag{0.19}$$

$$= -2\frac{(m\alpha)^3}{\hbar^4}\int_{0}^{\infty}e^{-2m\alpha x/\hbar^2}dx + 2\left(\frac{m\alpha}{\hbar}\right)^2 \tag{0.20}$$

$$= \left(\frac{m\alpha}{\hbar}\right)^2 \tag{0.21}$$

Going from the second to the third line, we used the property of the delta function

$$\int_{-\infty}^{\infty}f(x)\delta(x)dx = f(0) \tag{0.22}$$

The uncertainty principlefor the bound state of the delta function potential is therefore

$$\sigma_x\sigma_p = \sqrt{\left\langle x^2\right\rangle\left\langle p^2\right\rangle} \tag{0.23}$$

$$= \frac{\hbar}{\sqrt{2}} \tag{0.24}$$