

DELTA FUNCTION WELL: BOUND STATE - UNCERTAINTY PRINCIPLE

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.25.

We've seen that the bound state of a particle in a delta function potential well has the wave function

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2} \quad (1)$$

We can work out the mean values of position and momentum, and of their squares, in the usual way.

For $\langle x \rangle$, since the wave function is even and the function $f(x) = x$ is odd, we have

$$\langle x \rangle = \int_{-\infty}^{\infty} x\psi^2(x)dx \quad (2)$$

$$= 0 \quad (3)$$

since the integrand is odd.

For the mean square position, we get

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2\psi^2(x)dx \quad (4)$$

$$= \frac{m\alpha}{\hbar^2} \int_{-\infty}^{\infty} x^2 e^{-2m\alpha|x|/\hbar^2} dx \quad (5)$$

$$= \frac{2m\alpha}{\hbar^2} \int_0^{\infty} x^2 e^{-2m\alpha x/\hbar^2} dx \quad (6)$$

$$= \frac{\hbar^4}{2m^2\alpha^2} \quad (7)$$

where we used software to do the integral.

Now for the momentum. We get

$$\langle p \rangle = \frac{\hbar}{i} \frac{m\alpha}{\hbar^2} \int_{-\infty}^{\infty} e^{-m\alpha|x|/\hbar^2} \frac{d}{dx} e^{-m\alpha|x|/\hbar^2} dx \quad (8)$$

We can split this integral into two parts joined at the origin.

$$\frac{d}{dx} e^{-m\alpha|x|/\hbar^2} = \begin{cases} -\frac{m\alpha}{\hbar^2} e^{-m\alpha x/\hbar^2} & x > 0 \\ \frac{m\alpha}{\hbar^2} e^{m\alpha x/\hbar^2} & x < 0 \end{cases} \quad (9)$$

The derivative is therefore an odd function. Since the original wave function is even, the product of the two is odd, so $\langle p \rangle = 0$.

Calculating $\langle p^2 \rangle$ is a bit trickier, since the derivative above is discontinuous at the origin. If we define the step function

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \quad (10)$$

we can write the derivative above as

$$\frac{d}{dx} e^{-m\alpha|x|/\hbar^2} = -\frac{m\alpha}{\hbar^2} e^{-m\alpha|x|/\hbar^2} (2H(x) - 1) \quad (11)$$

We've seen that the derivative of the step function can be taken as the delta function

$$\frac{dH}{dx} = \delta(x) \quad (12)$$

so the second derivative of the wave function is

$$\frac{d^2}{dx^2} \psi(x) = \frac{\sqrt{m\alpha}}{\hbar} \frac{d^2}{dx^2} e^{-m\alpha|x|/\hbar^2} \quad (13)$$

$$= -\frac{(m\alpha)^{3/2}}{\hbar^3} \frac{d}{dx} \left[e^{-m\alpha|x|/\hbar^2} (2H(x) - 1) \right] \quad (14)$$

$$= \frac{(m\alpha)^{5/2}}{\hbar^5} e^{-m\alpha|x|/\hbar^2} (2H(x) - 1)^2 - 2 \frac{(m\alpha)^{3/2}}{\hbar^3} e^{-m\alpha|x|/\hbar^2} \delta(x) \quad (15)$$

Here we use the derivative of $e^{-m\alpha|x|/\hbar^2}$ which, if applied separately for $x < 0$ and $x > 0$, gives

$$\frac{d}{dx}e^{-m\alpha|x|/\hbar^2} = \begin{cases} \frac{m\alpha}{\hbar^2}e^{-m\alpha|x|/\hbar^2} & \text{for } x < 0 \\ -\frac{m\alpha}{\hbar^2}e^{-m\alpha|x|/\hbar^2} & \text{for } x > 0 \end{cases} \quad (16)$$

$$= -\frac{m\alpha}{\hbar^2}(2H(x) - 1)e^{-m\alpha|x|/\hbar^2} \quad (17)$$

Observing that $(2H(x) - 1)^2 = 1$ everywhere, we get

$$\frac{d^2}{dx^2}\psi(x) = e^{-m\alpha|x|/\hbar^2} \left[\frac{(m\alpha)^{5/2}}{\hbar^5} - 2\frac{(m\alpha)^{3/2}}{\hbar^3}\delta(x) \right] \quad (18)$$

After all this we can now calculate $\langle p^2 \rangle$:

$$\langle p^2 \rangle = \left(\frac{\hbar}{i} \right)^2 \int_{-\infty}^{\infty} \psi(x) \frac{d^2}{dx^2} \psi(x) dx \quad (19)$$

$$= -\hbar^2 \frac{\sqrt{m\alpha}}{\hbar} \int_{-\infty}^{\infty} e^{-2m\alpha|x|/\hbar^2} \left[\frac{(m\alpha)^{5/2}}{\hbar^5} - 2\frac{(m\alpha)^{3/2}}{\hbar^3}\delta(x) \right] dx \quad (20)$$

$$= -\frac{(m\alpha)^3}{\hbar^4} \int_{-\infty}^{\infty} e^{-2m\alpha|x|/\hbar^2} dx + 2 \left(\frac{m\alpha}{\hbar} \right)^2 \quad (21)$$

$$= -2\frac{(m\alpha)^3}{\hbar^4} \int_0^{\infty} e^{-2m\alpha x/\hbar^2} dx + 2 \left(\frac{m\alpha}{\hbar} \right)^2 \quad (22)$$

$$= \left(\frac{m\alpha}{\hbar} \right)^2 \quad (23)$$

Going from the second to the third line, we used the property of the delta function

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0) \quad (24)$$

The uncertainty principle for the bound state of the delta function potential is therefore

$$\sigma_x \sigma_p = \sqrt{\langle x^2 \rangle \langle p^2 \rangle} \quad (25)$$

$$= \frac{\hbar}{\sqrt{2}} \quad (26)$$

COMMENTS

Remark 1. From hladacpravdy; Nov 4, 2017 2:45 PM

Equation 21: $= (m * \alpha / \hbar)^2$

From this follows: $\sigma_p = \sqrt{\quad} = m * \alpha / \hbar$

Unit analysis: $m \dots \text{kg} \alpha \dots \text{J} \hbar \dots \text{Js} m * \alpha / \hbar \dots \text{kg/s}$

SI unit for momentum is kg m/s. Where I have done mistake ?

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α has units of Energy \times Length - the exponent in 1 must be dimensionless.