FINITE SQUARE WELL - NORMALIZATION

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.30.

The finite square well has the potential

$$V(x)=\begin{cases}0 & x<-a\\ -V_0 & -a\le x\le a\\ 0 & x>a\end{cases} \tag{1}$$

where V_0 is a positive constant energy, and a is a constant location on the x axis.

We've seen that since the potential is an even function, we can look for solutions of the Schrödinger equation that are either even or odd. In the even function case, we've seen that the solution has the form

$$\psi(x)=\begin{cases}Be^{\kappa x} & x<-a\\ D\cos(\mu x) & -a\le x\le a\\ Be^{-\kappa x} & x>a\end{cases} \tag{2}$$

Imposing boundary conditions gives us the relations

$$Be^{-\kappa a} = D\cos(\mu a) \tag{3}$$
$$-\kappa Be^{-\kappa a} = -\mu D\sin(\mu a) \tag{4}$$

from which we get the relation between κ and μ:

$$\kappa=\mu\tan(\mu a) \tag{5}$$

To complete the analysis, we need to normalize the wave function. Since it's an even function, the integral from 0 to ∞ of the square modulus is half the integral from $-\infty$ to ∞, so we get:

$$D^2\int_0^a\cos^2\mu x dx+B^2\int_a^\infty e^{-2\kappa x}dx=\frac{1}{2} \tag{6}$$

Doing the integrals gives us

$$\frac{1}{2\mu}D^2\left(\sin(\mu a)\cos(\mu a)+\mu a\right)+\frac{1}{2\kappa}B^2e^{-2\kappa a}=\frac{1}{2} \tag{7}$$

From the first boundary condition above, we have

$$B^2e^{-2\kappa a}=D^2\cos^2(\mu a) \tag{8}$$

Solving these last two equations, we get

$$\begin{aligned} D^2 &= \frac{\kappa\mu}{\mu\cos^2(\mu a)+\kappa\sin(\mu a)\cos(\mu a)+\kappa\mu a} && (9)\\ B^2 &= \frac{\kappa\mu\cos^2(\mu a)\,e^{2\kappa a}}{\mu\cos^2(\mu a)+\kappa\sin(\mu a)\cos(\mu a)+\kappa\mu a} && (10)\end{aligned}$$

Using the relation between κ and μ along with the trig identity $\cos^2 x+\sin^2 x=1$ we get

$$\begin{aligned} D^2 &= \frac{\kappa\mu}{\mu\cos^2(\mu a)+\mu\tan(\mu a)\sin(\mu a)\cos(\mu a)+\kappa\mu a} && (11)\\ &= \frac{\kappa}{1+\kappa a} && (12)\\ D &= \sqrt{\frac{\kappa}{1+\kappa a}} && (13)\\ B &= \sqrt{\frac{\kappa}{1+\kappa a}}\cos(\mu a)\,e^{\kappa a} && (14)\end{aligned}$$

While we're at it, we can also normalize the odd solution. Here, the solution has the form

$$\psi(x)=\begin{cases} Be^{\kappa x} & x<-a\\ C\sin(\mu x) & -a\le x\le a\\ -Be^{-\kappa x} & x>a\end{cases} \tag{15}$$

The boundary conditions give

$$\begin{aligned} Be^{-\kappa a} &= -C\sin(\mu a) && (16)\\ \kappa Be^{-\kappa a} &= \mu C\cos(\mu a) && (17)\end{aligned}$$

with the resulting relation between κ and μ:

$$\frac{1}{\kappa}=-\frac{1}{\mu}\tan(\mu a) \tag{18}$$

The normalization integral is

$$C^2 \int_0^a \sin^2 \mu x dx + B^2 \int_a^\infty e^{-2\kappa x} dx = \frac{1}{2} \tag{19}$$

which gives

$$-\frac{1}{2\mu} C^2 (\sin(\mu a)\cos(\mu a) - \mu a) + \frac{1}{2\kappa} B^2 e^{-2ka} = \frac{1}{2} \tag{20}$$

The first boundary condition gives

$$B^2 e^{-2\kappa a} = C^2 \sin^2(\mu a) \tag{21}$$

Solving these two equations gives

$$C^2 = \frac{\kappa\mu}{\mu \sin^2(\mu a) - \kappa \sin(\mu a)\cos(\mu a) + \kappa\mu a} \tag{22}$$

$$B^2 = \frac{\kappa\mu \sin^2(\mu a) e^{2\kappa a}}{\mu \sin^2(\mu a) - \kappa \sin(\mu a)\cos(\mu a) + \kappa\mu a} \tag{23}$$

Rewriting the relation between κ and μ we get

$$\kappa = -\mu \frac{\cos(\mu a)}{\sin(\mu a)} \tag{24}$$

and inserting this into the solutions gives

$$C^2 = \frac{\kappa}{1 + \kappa a} \tag{25}$$

$$B^2 = \frac{\kappa}{1 + \kappa a} \sin^2(\mu a) e^{2\kappa a} \tag{26}$$

The final result is then

$$C = \sqrt{\frac{\kappa}{1 + \kappa a}} \tag{27}$$

$$B = -\sqrt{\frac{\kappa}{1 + \kappa a}} \sin(\mu a) e^{\kappa a} \tag{28}$$

where we've taken the negative root of B^2 in order to satisfy the original boundary conditions above (we could also have taken C with a negative sign and B with a positive sign).