

DELTA FUNCTION WELL AS LIMIT OF FINITE SQUARE WELL

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Post date: 5 Aug 2012.

Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.31.

The finite square well has the potential

$$(1) \quad V(x) = \begin{cases} 0 & x < -a \\ -V_0 & -a \leq x \leq a \\ 0 & x > a \end{cases}$$

where V_0 is a positive constant energy, and a is a constant location on the x axis.

The delta function potential $V = -\alpha\delta(x)$ can be thought of as the limit of the finite square well as $a \rightarrow 0$ and $V_0 \rightarrow \infty$ in such a way that the area of the rectangle in the well is a constant. That is, the integral of the potential is the same in both cases, so that

$$(2) \quad 2aV_0 = \alpha$$

The energies of the bound states for the even solution of the finite square well are given by

$$(3) \quad \tan z = \sqrt{\frac{z_0^2}{z^2} - 1}$$

where

$$(4) \quad z_0 \equiv \frac{a}{\hbar} \sqrt{2mV_0}$$

$$(5) \quad z \equiv \frac{a}{\hbar} \sqrt{2m(E + V_0)}$$

Substituting for V_0 we get

$$(6) \quad z_0 = \frac{1}{\hbar} \sqrt{ma\alpha}$$

$$(7) \quad z = \frac{1}{\hbar} \sqrt{2ma^2E + ma\alpha}$$

As $a \rightarrow 0$, $z_0/z \rightarrow 1$, so $\tan z$ becomes very small. In this limit, $\tan z \approx z$, so we can approximate equation 3 by

$$(8) \quad z = \sqrt{\frac{z_0^2}{z^2} - 1}$$

$$(9) \quad \frac{1}{\hbar} \sqrt{2ma^2E + ma\alpha} = \sqrt{\frac{\alpha}{2aE + \alpha} - 1}$$

$$(10) \quad \frac{1}{\hbar^2} (2ma^2E + ma\alpha) = \frac{\alpha}{2aE + \alpha} - 1$$

$$(11) \quad (2ma^2E + ma\alpha)(2aE + \alpha) = -2aE\hbar^2$$

If we retain only the term in a (discarding higher powers of a), we get

$$(12) \quad ma\alpha^2 = -2aE\hbar^2$$

$$(13) \quad E = -\frac{m\alpha^2}{2\hbar^2}$$

This is the energy we found earlier when analyzing the delta function well.

We can do a similar analysis for the scattering states. The transmission coefficient for the finite square well is

$$(14) \quad T^{-1} = 1 + \frac{V_0^2}{4E(E + V_0)} \sin^2 \left(\frac{2a}{\hbar} \sqrt{2m(E + V_0)} \right)$$

For small a we can use the approximation $\sin x \approx x$ and we get

$$(15) \quad T^{-1} \approx 1 + \frac{2mV_0^2 a^2}{\hbar^2 E}$$

Substituting for V_0 gives

$$(16) \quad T^{-1} \approx 1 + \frac{m\alpha^2}{2\hbar^2 E}$$

$$(17) \quad T = \frac{1}{1 + m\alpha^2/2\hbar^2 E}$$

This is the same formula we obtained when analyzing the delta function potential directly.

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