

DELTA FUNCTION WELL AS LIMIT OF FINITE SQUARE WELL

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.31.

The finite square well has the potential

$$V(x) = \begin{cases} 0 & x < -a \\ -V_0 & -a \leq x \leq a \\ 0 & x > a \end{cases} \quad (1)$$

where V_0 is a positive constant energy, and a is a constant location on the x axis.

The delta function potential $V = -\alpha\delta(x)$ can be thought of as the limit of the finite square well as $a \rightarrow 0$ and $V_0 \rightarrow \infty$ in such a way that the area of the rectangle in the well is a constant. That is, the integral of the potential is the same in both cases, so that

$$2aV_0 = \alpha \quad (2)$$

The energies of the bound states for the even solution of the finite square well are given by

$$\tan z = \sqrt{\frac{z_0^2}{z^2} - 1} \quad (3)$$

where

$$z_0 \equiv \frac{a}{\hbar} \sqrt{2mV_0} \quad (4)$$

$$z \equiv \frac{a}{\hbar} \sqrt{2m(E + V_0)} \quad (5)$$

Substituting for V_0 we get

$$z_0 = \frac{1}{\hbar} \sqrt{ma\alpha} \quad (6)$$

$$z = \frac{1}{\hbar} \sqrt{2ma^2E + ma\alpha} \quad (7)$$

As $a \rightarrow 0$, $z_0/z \rightarrow 1$, so $\tan z$ becomes very small. In this limit, $\tan z \approx z$, so we can approximate equation 3 by

$$z = \sqrt{\frac{z_0^2}{z^2} - 1} \quad (8)$$

$$\frac{1}{\hbar} \sqrt{2ma^2E + ma\alpha} = \sqrt{\frac{\alpha}{2aE + \alpha} - 1} \quad (9)$$

$$\frac{1}{\hbar^2} (2ma^2E + ma\alpha) = \frac{\alpha}{2aE + \alpha} - 1 \quad (10)$$

$$(2ma^2E + ma\alpha)(2aE + \alpha) = -2aE\hbar^2 \quad (11)$$

If we retain only the term in a (discarding higher powers of a), we get

$$ma\alpha^2 = -2aE\hbar^2 \quad (12)$$

$$E = -\frac{m\alpha^2}{2\hbar^2} \quad (13)$$

This is the energy we found earlier when analyzing the delta function well.

We can do a similar analysis for the scattering states. The transmission coefficient for the finite square well is

$$T^{-1} = 1 + \frac{V_0^2}{4E(E + V_0)} \sin^2 \left(\frac{2a}{\hbar} \sqrt{2m(E + V_0)} \right) \quad (14)$$

For small a we can use the approximation $\sin x \approx x$ and we get

$$T^{-1} \approx 1 + \frac{2mV_0^2 a^2}{\hbar^2 E} \quad (15)$$

Substituting for V_0 gives

$$T^{-1} \approx 1 + \frac{m\alpha^2}{2\hbar^2 E} \quad (16)$$

$$T = \frac{1}{1 + m\alpha^2/2\hbar^2 E} \quad (17)$$

This is the same formula we obtained when analyzing the delta function potential directly.

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