FINITE SQUARE WELL - SCATTERING

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.32.
The finite square well has the potential

$$V(x)=\begin{cases}0 & x<-a\\ -V_0 & -a\le x\le a\\ 0 & x>a\end{cases} \tag{1}$$

where V_0 is a positive constant energy, and a is a constant location on the x axis.

We've analyzed the bound states of this potential, so we can have a look now at the scattering states, where $E>0$. Following a similar procedure to that for the scattering states with the delta function potential, we solve the Schrödinger equation in the three regions of the x axis. For $x<-a$ we have

$$\psi(x)=Ae^{ikx}+Be^{-ikx} \tag{2}$$

where

$$k=\frac{\sqrt{2mE}}{\hbar} \tag{3}$$

For $-a<x<a$ we have

$$\psi(x)=C\sin(\mu x)+D\cos(\mu x) \tag{4}$$

where

$$\mu=\frac{\sqrt{2m(E+V_0)}}{\hbar} \tag{5}$$

For $x>a$ we assume as usual that the only incoming wave is from negative x so the only wave in this region is moving to the right (outgoing). Thus

$$\psi(x)=Fe^{ikx} \tag{6}$$

Since the potential is finite everywhere, both the wave function and its derivative must be continuous everywhere. At $x=-a$ these two conditions yield

$$\begin{aligned} Ae^{-ika}+Be^{ika} &= -C\sin(\mu a)+D\cos(\mu a) && (7)\\ ik\left(Ae^{-ika}-Be^{ika}\right) &= \mu\left[C\cos(\mu a)+D\sin(\mu a)\right] && (8)\end{aligned}$$

At $x=a$ we have

$$\begin{aligned} C\sin(\mu a)+D\cos(\mu a) &= Fe^{ika} && (9)\\ \mu\left[C\cos(\mu a)-D\sin(\mu a)\right] &= ikFe^{ika} && (10)\end{aligned}$$

We have four linear equations in five unknowns, so we can solve for four of the constants in terms of the remaining one. This is most easily done using software, such as Maple's 'solve' command. The results are

$$\begin{aligned} B &= \frac{e^{-2ika}\left(k^2-\mu^2\right)\sin(2\mu a)}{\sin(2\mu a)(k^2+\mu^2)+2i\mu k\cos(2\mu a)}A && (11)\\ C &= \frac{ke^{-ika}\left(k\sin(\mu a)-i\mu\cos(\mu a)\right)}{\cos^2(\mu a)(k^2-\mu^2)-k^2}A && (12)\\ D &= \frac{ke^{-ika}\left(k\cos(\mu a)+i\mu\sin(\mu a)\right)}{\cos^2(\mu a)(k^2-\mu^2)+\mu^2}A && (13)\\ F &= \frac{2\mu ke^{-2ika}}{2\mu k\cos(2\mu a)-i\sin(2\mu a)(k^2+\mu^2)}A && (14)\end{aligned}$$

The transmission coefficient is

$$\begin{aligned} T &= \frac{|F|^2}{|A|^2} && (15)\\ &= \frac{4k^2\mu^2}{\sin^2(2\mu a)\left[k^4+\mu^4-2\mu^2k^2\right]+4\mu^2k^2} && (16)\\ &= \frac{1}{1+(k^2-\mu^2)^2\sin^2(2\mu a)/4k^2\mu^2} && (17)\end{aligned}$$

The reciprocal of T is

$$T^{-1}=1+\frac{\left(k^{2}-\mu^{2}\right)^{2}\sin^{2}\left(2\mu a\right)}{4k^{2}\mu^{2}} \tag{18}$$

We can write this in terms of the original physical parameters by substituting for k and μ. We get

$$T^{-1}=1+\frac{V_{0}^{2}}{4E\left(E+V_{0}\right)}\sin^{2}\left(\frac{2a}{\hbar}\sqrt{2m\left(E+V_{0}\right)}\right) \tag{19}$$

The transmission probability becomes 1 (that is, there is no reflection) whenever the sine term is zero, which occurs at

$$\frac{2a}{\hbar}\sqrt{2m\left(E+V_{0}\right)}=n\pi \tag{20}$$

This phenomenon occurs in the Ramsauer-Townsend effect, which involves the scattering of electrons off atoms of inert gases. Classical physics predicts that the number of electrons scattered should increase monotonically with their energy, but in fact a minimum is observed for certain electron energies. A model in which the inert gas atom is treated as a finite square well provides a simplified explanation of the effect.

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