# FINITE SQUARE BARRIER - SCATTERING

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.33.

Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 5.4, Exercise 5.4.2 (a).

We've analyzed the scattering problem in the finite square well, and we can use similar techniques to analyze a finite square barrier, which has the potential

$$V(x)=\begin{cases}0 & x<-a\\ V_0 & -a\le x\le a\\ 0 & x>a\end{cases} \tag{0.1}$$

where $V_0$ is a positive constant energy, and $a$ is a constant location on the $x$ axis.

There are three distinct cases here:

(1) Energy below the barrier: $0\le E<V_0$
(2) Energy exactly equal to the barrier: $E=V_0$
(3) Energy greater than the barrier: $E>V_0$

In all three cases, the wave function away from the barrier on either side has the same form; it is only within the barrier that the three cases differ. We'll consider first the case where $0\le E<V_0$.

In this case, the Schrödinger equation within the barrier is

$$-\frac{\hbar^2}{2m}\psi''+V_0\psi = E\psi \tag{0.2}$$

$$\psi'' = \mu^2\psi \tag{0.3}$$

where

$$\mu=\frac{\sqrt{2m(V_0-E)}}{\hbar} \tag{0.4}$$

This has solution

$$\psi(x) = Ce^{\mu x} + De^{-\mu x} \tag{0.5}$$

Outside the barrier, the Schrödinger equation is

$$\psi'' = E\psi \tag{0.6}$$

Outside the barrier on the left, the solution is the sum of travelling waves (assuming particles are incident from the left only), while to the right we have right propagating waves only. Thus

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < -a \\ Fe^{ikx} & x > a \end{cases} \tag{0.7}$$

where

$$k = \frac{\sqrt{2mE}}{\hbar} \tag{0.8}$$

Since the potential is finite everywhere, both $\psi$ and $\psi'$ are continuous everywhere, which gives us four boundary conditions.

At $x = -a$ we have

$$Ae^{-ika} + Be^{ika} = Ce^{-\mu a} + De^{\mu a} \tag{0.9}$$

$$ik\left(Ae^{-ika} - Be^{ika}\right) = \mu\left(Ce^{-\mu a} - De^{\mu a}\right) \tag{0.10}$$

At $x = a$:

$$Ce^{\mu a} + De^{-\mu a} = Fe^{ika} \tag{0.11}$$

$$\mu\left(Ce^{\mu a} - De^{-\mu a}\right) = ikFe^{ika} \tag{0.12}$$

We can solve these linear equations to get the other four constants in terms of $A$. The results are

$$B = \frac{e^{-2ika}\left(k^2+\mu^2\right)\left(e^{2\mu a}-e^{-2\mu a}\right)}{2ik\mu\left(e^{2\mu a}+e^{-2\mu a}\right)+\left(k^2-\mu^2\right)\left(e^{2\mu a}-e^{-2\mu a}\right)}A \tag{0.13}$$

$$= \frac{e^{-2ika}\left(k^2+\mu^2\right)\sinh(2\mu a)}{2ik\mu\cosh(2\mu a)+\left(k^2-\mu^2\right)\sinh(2\mu a)}A \tag{0.14}$$

$$C = \frac{e^{-a\mu-iak}\left(-k^2+k\mu i\right)}{2ik\mu\cosh(2\mu a)+\left(k^2-\mu^2\right)\sinh(2\mu a)}A \tag{0.15}$$

$$D = \frac{e^{-a\mu-iak}\left(k^2+k\mu i\right)}{2ik\mu\cosh(2\mu a)+\left(k^2-\mu^2\right)\sinh(2\mu a)}A \tag{0.16}$$

$$F = \frac{2e^{-2ika}k\mu i}{2ik\mu\cosh(2\mu a)+\left(k^2-\mu^2\right)\sinh(2\mu a)}A \tag{0.17}$$

From here we can get the transmission coefficient as

$$T = \frac{|F|^2}{|A|^2} \tag{0.18}$$

$$= \frac{4\mu^2k^2}{\left[\mu^4+2\mu^2k^2+k^4\right]\sinh^2(2\mu a)+4\mu^2k^2} \tag{0.19}$$

$$= \frac{1}{\left(\mu^2+k^2\right)^2\sinh^2(2\mu a)/4\mu^2k^2+1} \tag{0.20}$$

The reciprocal of $T$ is then, substituting to get the physical quantities back:

$$T^{-1} = 1+\frac{\left(\mu^2+k^2\right)^2\sinh^2(2\mu a)}{4\mu^2k^2} \tag{0.21}$$

$$= 1+\frac{V_0^2}{4E\left(V_0-E\right)}\sinh^2\left(\frac{2a}{\hbar}\sqrt{2m\left(V_0-E\right)}\right) \tag{0.22}$$

The second case is where $E=V_0$. In this case, the outer two solutions are the same as before, but in the barrier region we have

$$\psi''=0 \tag{0.23}$$

which has the solution

$$\psi = Cx+D \tag{0.24}$$

Applying the boundary conditions we have, at $x=-a$

$$Ae^{-ika}+Be^{ika} = -Ca+D \tag{0.25}$$

$$ik\left(Ae^{-ika}-Be^{ika}\right) = C \tag{0.26}$$

At $x=a$ we have

$$Ca+D = Fe^{ika} \tag{0.27}$$

$$C = ikFe^{ika} \tag{0.28}$$

Solving these equations we get

$$B = \frac{kae^{-2ika}}{ka+i}A \tag{0.29}$$

$$C = \frac{ke^{-ika}}{-ka-i}A \tag{0.30}$$

$$D = e^{-ika}A \tag{0.31}$$

$$F = \frac{e^{-2ika}}{1-ika}A \tag{0.32}$$

In this case the transmission coefficient is

$$T = \frac{|F|^2}{|A|^2} \tag{0.33}$$

$$= \frac{1}{1+k^2a^2} \tag{0.34}$$

$$= \frac{1}{1+2mEa^2/\hbar^2} \tag{0.35}$$

Finally, for $E>V_0$ the Schrödinger equation within the barrier is

$$-\frac{\hbar^2}{2m}\psi'' = (E-V_0)\psi \tag{0.36}$$

$$\psi'' = -\frac{2m(E-V_0)}{\hbar^2}\psi \tag{0.37}$$

$$= -\lambda^2\psi \tag{0.38}$$

where

$$\lambda=\frac{\sqrt{2m(E-V_0)}}{\hbar} \tag{0.39}$$

The solution within the barrier is thus

$$\psi(x)=Ce^{i\lambda x}+De^{-i\lambda x} \tag{0.40}$$

Boundary conditions give at $x=-a$

$$\begin{aligned}
Ae^{-ika}+Be^{ika} &= Ce^{-i\lambda a}+De^{i\lambda a} && (0.41)\\
ik\left(Ae^{-ika}-Be^{ika}\right) &= i\lambda\left(Ce^{-i\lambda a}-De^{i\lambda a}\right) && (0.42)
\end{aligned}$$

At $x=a$:

$$\begin{aligned}
Ce^{i\lambda a}+De^{-i\lambda a} &= Fe^{ika} && (0.43)\\
i\lambda\left(Ce^{i\lambda a}-De^{-i\lambda a}\right) &= ikFe^{ika} && (0.44)
\end{aligned}$$

Solving these four equations gives

$$\begin{aligned}
B &= \frac{e^{-2ika}\left(k^2-\lambda^2\right)\sin(2\lambda a)}{(k^2+\lambda^2)\sin(2\lambda a)+2i\lambda k\cos(2\lambda a)}A && (0.45)\\
C &= \frac{e^{-ia(\lambda+k)}(\lambda+k)k}{-i(k^2+\lambda^2)\sin(2\lambda a)+2\lambda k\cos(2\lambda a)}A && (0.46)\\
D &= \frac{e^{-ia(k-\lambda)}(k-\lambda)k}{i(k^2+\lambda^2)\sin(2\lambda a)-2\lambda k\cos(2\lambda a)}A && (0.47)\\
F &= \frac{2k\lambda e^{-2ika}}{-i(k^2+\lambda^2)\sin(2\lambda a)+2\lambda k\cos(2\lambda a)}A && (0.48)
\end{aligned}$$

The transmission coefficient is

$$\begin{aligned}
T &= \frac{|F|^2}{|A|^2} && (0.49)\\
&= \frac{4\lambda^2k^2}{(\lambda^4-2\lambda^2k^2+k^4)\sin^2(2\lambda a)+4\lambda^2k^2} && (0.50)\\
&= \frac{1}{1+(\lambda^2-k^2)^2\sin^2(2\lambda a)/4\lambda^2k^2} && (0.51)
\end{aligned}$$

The reciprocal is

$$T^{-1} = 1+\frac{\left(\lambda^{2}-k^{2}\right)^{2}\sin^{2}\left(2\lambda a\right)}{4\lambda^{2}k^{2}} \tag{0.52}$$

$$= 1+\frac{V_{0}^{2}}{4E\left(E-V_{0}\right)}\sin^{2}\left(\frac{2a}{\hbar}\sqrt{2m\left(E-V_{0}\right)}\right) \tag{0.53}$$

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