FINITE STEP POTENTIAL - SCATTERING

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.34.

A variant of the finite square well is the finite step, which has the potential

$$V(x)=\begin{cases}0 & x\le 0\\ V_0 & x>0\end{cases} \tag{1}$$

where V_0 is a positive constant energy.

There are two distinct cases here:

(1) Energy below the barrier: $0\le E\le V_0$

(2) Energy greater than the barrier: $E>V_0$

We'll consider first the case where $0\le E\le V_0$.

In this case, the Schrödinger equation for $x>0$ is

$$-\frac{\hbar^2}{2m}\psi''+V_0\psi = E\psi \tag{2}$$

$$\psi'' = \mu^2\psi \tag{3}$$

where

$$\mu=\frac{\sqrt{2m(V_0-E)}}{\hbar} \tag{4}$$

This has solution

$$\psi(x)=Ce^{\mu x}+De^{-\mu x} \tag{5}$$

To keep the solution finite as $x\to\infty$ we must have $C=0$ so the solution is an exponentially decaying wave function:

$$\psi(x)=De^{-\mu x} \tag{6}$$

To the left of the barrier, the Schrödinger equation is

$$\psi''=-\frac{2mE}{\hbar^2}\psi \tag{7}$$

Assuming particles coming in from the left, we have

$$\psi(x) = Ae^{ikx} + Be^{-ikx} \tag{8}$$

where

$$k = \frac{\sqrt{2mE}}{\hbar} \tag{9}$$

Since the potential is finite everywhere, both ψ and ψ' are continuous everywhere, which gives us two boundary conditions at $x = 0$.

$$\begin{aligned} A + B &= D &(10)\\ ik(A - B) &= -\mu D &(11)\end{aligned}$$

This has solution

$$\begin{aligned} B &= \frac{ik+\mu}{ik-\mu}A &(12)\\ D &= \frac{2ik}{ik-\mu}A &(13)\end{aligned}$$

The reflection coefficient is

$$\begin{aligned} R &= \frac{|B|^2}{|A|^2} &(14)\\ &= 1 &(15)\end{aligned}$$

That is, the probability of an incoming particle being reflected is 1. This is because the wave function for $x > 0$ is exponentially decaying, so the probability of a particle reaching infinity is zero, thus no particles can be transmitted.

For $E > V_0$ the Schrödinger equation for $x > 0$ is

$$\begin{aligned} -\frac{\hbar^2}{2m}\psi'' + V_0\psi &= E\psi &(16)\\ \psi'' &= -\kappa^2\psi &(17)\end{aligned}$$

where

$$\kappa = \frac{\sqrt{2m(E-V_0)}}{\hbar} \tag{18}$$

We now get travelling wave solutions instead of exponentially decaying ones:

$$\psi(x) = Ce^{i\kappa x} + De^{-i\kappa x} \tag{19}$$

Assuming incoming particles arrive only from the left, we can set $D = 0$. Applying the boundary conditions, we get

$$\begin{aligned} A+B &= C && (20) \\ ik(A-B) &= i\kappa C && (21) \end{aligned}$$

with solutions

$$\begin{aligned} B &= \frac{k-\kappa}{k+\kappa}A && (22) \\ C &= \frac{2k}{k+\kappa}A && (23) \end{aligned}$$

In this case, the reflection coefficient is

$$\begin{aligned} R &= \frac{|B|^2}{|A|^2} && (24) \\ &= \left(\frac{k-\kappa}{k+\kappa}\right)^2 && (25) \end{aligned}$$

Substituting the expressions for k and κ we get

$$R = \left[\frac{E-\sqrt{E(E-V_0)}}{E+\sqrt{E(E-V_0)}}\right]^2 \tag{26}$$

From this we can get the transmission coefficient

$$\begin{aligned} T &= 1-R && (27) \\ &= \frac{4E^{3/2}\sqrt{E-V_0}}{\left(E+\sqrt{E(E-V_0)}\right)^2} && (28) \end{aligned}$$

Note that this is *not* equal to $|C|^2/|A|^2 = 4E^2/\left(E+\sqrt{E(E-V_0)}\right)^2$. Have we done something wrong?

The answer lies in the fact that the wave for $x > 0$ is not the same as the wave for $x < 0$, since the net energy on the right is $E - V_0$ while on the left it is just E. One way of looking at it is in terms of the probability current for the free particle. The probability current must be conserved; this is just a way of saying that particles cannot vanish into, nor arise from, thin air. Since the probability current for a free particle with stationary state

$$\Psi(x,t) = Ae^{ikx}e^{-i\hbar k^2 t/2m} \tag{29}$$

is

$$J = \frac{\hbar k}{m}|A|^2 \tag{30}$$

the conservation law implies, for the case of the step potential

$$\frac{\hbar k}{m}\left[|A|^2 - |B|^2\right] = \frac{\hbar\kappa}{m}|C|^2 \tag{31}$$

That is, the influx of particles from the left minus the reflected beam must equal the transmitted beam. Dividing through by $\frac{\hbar k}{m}|A|^2$ we get

$$\frac{|B|^2}{|A|^2} + \frac{\kappa}{k}\frac{|C|^2}{|A|^2} = 1 \tag{32}$$

The first term is the reflection coefficient we calculated in 26. The second term is the transmission coefficient, which works out to

$$\begin{aligned} T &= \sqrt{\frac{E-V_0}{E}}\frac{|C|^2}{|A|^2} && (33)\\ &= \sqrt{\frac{E-V_0}{E}}\frac{4E^2}{\left(E+\sqrt{E(E-V_0)}\right)^2} && (34)\\ &= \frac{4E^{3/2}\sqrt{E-V_0}}{\left(E+\sqrt{E(E-V_0)}\right)^2} && (35) \end{aligned}$$

which is what we got earlier.

PINGBACKS

Pingback: Finite drop potential