

FINITE STEP POTENTIAL - SCATTERING

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.34.

A variant of the finite square well is the finite step, which has the potential

$$V(x) = \begin{cases} 0 & x \leq 0 \\ V_0 & x > 0 \end{cases} \quad (1)$$

where V_0 is a positive constant energy.

There are two distinct cases here:

(1) Energy below the barrier: $0 \leq E \leq V_0$

(2) Energy greater than the barrier: $E > V_0$

We'll consider first the case where $0 \leq E \leq V_0$.

In this case, the Schrödinger equation for $x > 0$ is

$$-\frac{\hbar^2}{2m}\psi'' + V_0\psi = E\psi \quad (2)$$

$$\psi'' = \mu^2\psi \quad (3)$$

where

$$\mu = \frac{\sqrt{2m(V_0 - E)}}{\hbar} \quad (4)$$

This has solution

$$\psi(x) = Ce^{\mu x} + De^{-\mu x} \quad (5)$$

To keep the solution finite as $x \rightarrow \infty$ we must have $C = 0$ so the solution is an exponentially decaying wave function:

$$\psi(x) = De^{-\mu x} \quad (6)$$

To the left of the barrier, the Schrödinger equation is

$$\psi'' = -\frac{2mE}{\hbar^2}\psi \quad (7)$$

Assuming particles coming in from the left, we have

$$\psi(x) = Ae^{ikx} + Be^{-ikx} \quad (8)$$

where

$$k = \frac{\sqrt{2mE}}{\hbar} \quad (9)$$

Since the potential is finite everywhere, both ψ and ψ' are continuous everywhere, which gives us two boundary conditions at $x = 0$.

$$A + B = D \quad (10)$$

$$ik(A - B) = -\mu D \quad (11)$$

This has solution

$$B = \frac{ik + \mu}{ik - \mu} A \quad (12)$$

$$D = \frac{2ik}{ik - \mu} A \quad (13)$$

The reflection coefficient is

$$R = \frac{|B|^2}{|A|^2} \quad (14)$$

$$= 1 \quad (15)$$

That is, the probability of an incoming particle being reflected is 1. This is because the wave function for $x > 0$ is exponentially decaying, so the probability of a particle reaching infinity is zero, thus no particles can be transmitted.

For $E > V_0$ the Schrödinger equation for $x > 0$ is

$$-\frac{\hbar^2}{2m}\psi'' + V_0\psi = E\psi \quad (16)$$

$$\psi'' = -\kappa^2\psi \quad (17)$$

where

$$\kappa = \frac{\sqrt{2m(E - V_0)}}{\hbar} \quad (18)$$

We now get travelling wave solutions instead of exponentially decaying ones:

$$\psi(x) = Ce^{i\kappa x} + De^{-i\kappa x} \quad (19)$$

Assuming incoming particles arrive only from the left, we can set $D = 0$. Applying the boundary conditions, we get

$$A + B = C \quad (20)$$

$$ik(A - B) = i\kappa C \quad (21)$$

with solutions

$$B = \frac{k - \kappa}{k + \kappa} A \quad (22)$$

$$C = \frac{2k}{k + \kappa} A \quad (23)$$

In this case, the reflection coefficient is

$$R = \frac{|B|^2}{|A|^2} \quad (24)$$

$$= \left(\frac{k - \kappa}{k + \kappa} \right)^2 \quad (25)$$

Substituting the expressions for k and κ we get

$$R = \left[\frac{E - \sqrt{E(E - V_0)}}{E + \sqrt{E(E - V_0)}} \right]^2 \quad (26)$$

From this we can get the transmission coefficient

$$T = 1 - R \quad (27)$$

$$= \frac{4E^{3/2}\sqrt{E - V_0}}{\left(E + \sqrt{E(E - V_0)}\right)^2} \quad (28)$$

Note that this is *not* equal to $|C|^2 / |A|^2 = 4E^2 / \left(E + \sqrt{E(E - V_0)}\right)^2$. Have we done something wrong?

The answer lies in the fact that the wave for $x > 0$ is not the same as the wave for $x < 0$, since the net energy on the right is $E - V_0$ while on the left

it is just E . One way of looking at it is in terms of the probability current for the free particle. The probability current must be conserved; this is just a way of saying that particles cannot vanish into, nor arise from, thin air. Since the probability current for a free particle with stationary state

$$\Psi(x,t) = Ae^{ikx}e^{-i\hbar k^2 t/2m} \quad (29)$$

is

$$J = \frac{\hbar k}{m} |A|^2 \quad (30)$$

the conservation law implies, for the case of the step potential

$$\frac{\hbar k}{m} [|A|^2 - |B|^2] = \frac{\hbar \kappa}{m} |C|^2 \quad (31)$$

That is, the influx of particles from the left minus the reflected beam must equal the transmitted beam. Dividing through by $\frac{\hbar k}{m} |A|^2$ we get

$$\frac{|B|^2}{|A|^2} + \frac{\kappa}{k} \frac{|C|^2}{|A|^2} = 1 \quad (32)$$

The first term is the reflection coefficient we calculated in 26. The second term is the transmission coefficient, which works out to

$$T = \sqrt{\frac{E - V_0}{E}} \frac{|C|^2}{|A|^2} \quad (33)$$

$$= \sqrt{\frac{E - V_0}{E}} \frac{4E^2}{\left(E + \sqrt{E(E - V_0)} \right)^2} \quad (34)$$

$$= \frac{4E^{3/2} \sqrt{E - V_0}}{\left(E + \sqrt{E(E - V_0)} \right)^2} \quad (35)$$

which is what we got earlier.

PINGBACKS

Pingback: Finite drop potential