

FINITE STEP POTENTIAL - SCATTERING

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.34.

A variant of the finite square well is the finite step, which has the potential

$$(1) \quad V(x) = \begin{cases} 0 & x \leq 0 \\ V_0 & x > 0 \end{cases}$$

where V_0 is a positive constant energy.

There are two distinct cases here:

(1) Energy below the barrier: $0 \leq E \leq V_0$

(2) Energy greater than the barrier: $E > V_0$

We'll consider first the case where $0 \leq E \leq V_0$.

In this case, the Schrödinger equation for $x > 0$ is

$$(2) \quad -\frac{\hbar^2}{2m}\psi'' + V_0\psi = E\psi$$

$$(3) \quad \psi'' = \mu^2\psi$$

where

$$(4) \quad \mu = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

This has solution

$$(5) \quad \psi(x) = Ce^{\mu x} + De^{-\mu x}$$

To keep the solution finite as $x \rightarrow \infty$ we must have $C = 0$ so the solution is an exponentially decaying wave function:

$$(6) \quad \psi(x) = De^{-\mu x}$$

To the left of the barrier, the Schrödinger equation is

$$(7) \quad \psi'' = -\frac{2mE}{\hbar^2} \psi$$

Assuming particles coming in from the left, we have

$$(8) \quad \psi(x) = Ae^{ikx} + Be^{-ikx}$$

where

$$(9) \quad k = \frac{\sqrt{2mE}}{\hbar}$$

Since the potential is finite everywhere, both ψ and ψ' are continuous everywhere, which gives us two boundary conditions at $x = 0$.

$$(10) \quad A + B = D$$

$$(11) \quad ik(A - B) = -\mu D$$

This has solution

$$(12) \quad B = \frac{ik + \mu}{ik - \mu} A$$

$$(13) \quad D = \frac{2ik}{ik - \mu} A$$

The reflection coefficient is

$$(14) \quad R = \frac{|B|^2}{|A|^2}$$

$$(15) \quad = 1$$

That is, the probability of an incoming particle being reflected is 1. This is because the wave function for $x > 0$ is exponentially decaying, so the probability of a particle reaching infinity is zero, thus no particles can be transmitted.

For $E > V_0$ the Schrödinger equation for $x > 0$ is

$$(16) \quad -\frac{\hbar^2}{2m} \psi'' + V_0 \psi = E \psi$$

$$(17) \quad \psi'' = -\kappa^2 \psi$$

where

$$(18) \quad \kappa = \frac{\sqrt{2m(E - V_0)}}{\hbar}$$

We now get travelling wave solutions instead of exponentially decaying ones:

$$(19) \quad \psi(x) = Ce^{i\kappa x} + De^{-i\kappa x}$$

Assuming incoming particles arrive only from the left, we can set $D = 0$. Applying the boundary conditions, we get

$$(20) \quad A + B = C$$

$$(21) \quad ik(A - B) = i\kappa C$$

with solutions

$$(22) \quad B = \frac{k - \kappa}{k + \kappa} A$$

$$(23) \quad C = \frac{2k}{k + \kappa} A$$

In this case, the reflection coefficient is

$$(24) \quad R = \frac{|B|^2}{|A|^2}$$

$$(25) \quad = \left(\frac{k - \kappa}{k + \kappa} \right)^2$$

Substituting the expressions for k and κ we get

$$(26) \quad R = \left[\frac{E - \sqrt{E(E - V_0)}}{E + \sqrt{E(E - V_0)}} \right]^2$$

From this we can get the transmission coefficient

$$\begin{aligned}
 (27) \quad T &= 1 - R \\
 (28) \quad &= \frac{4E^{3/2}\sqrt{E-V_0}}{\left(E + \sqrt{E(E-V_0)}\right)^2}
 \end{aligned}$$

Note that this is *not* equal to $|C|^2/|A|^2 = 4E^2/\left(E + \sqrt{E(E-V_0)}\right)^2$. Have we done something wrong?

The answer lies in the fact that the wave for $x > 0$ is not the same as the wave for $x < 0$, since the net energy on the right is $E - V_0$ while on the left it is just E . One way of looking at it is in terms of the probability current for the free particle. The probability current must be conserved; this is just a way of saying that particles cannot vanish into, nor arise from, thin air. Since the probability current for a free particle with stationary state

$$(29) \quad \Psi(x,t) = Ae^{ikx}e^{-i\hbar k^2 t/2m}$$

is

$$(30) \quad J = \frac{\hbar k}{m} |A|^2$$

the conservation law implies, for the case of the step potential

$$(31) \quad \frac{\hbar k}{m} \left[|A|^2 - |B|^2\right] = \frac{\hbar \kappa}{m} |C|^2$$

That is, the influx of particles from the left minus the reflected beam must equal the transmitted beam. Dividing through by $\frac{\hbar k}{m} |A|^2$ we get

$$(32) \quad \frac{|B|^2}{|A|^2} + \frac{\kappa}{k} \frac{|C|^2}{|A|^2} = 1$$

The first term is the reflection coefficient we calculated in 26. The second term is the transmission coefficient, which works out to

$$(33) \quad T = \sqrt{\frac{E - V_0}{E}} \frac{|C|^2}{|A|^2}$$

$$(34) \quad = \sqrt{\frac{E - V_0}{E}} \frac{4E^2}{\left(E + \sqrt{E(E - V_0)}\right)^2}$$

$$(35) \quad = \frac{4E^{3/2}\sqrt{E - V_0}}{\left(E + \sqrt{E(E - V_0)}\right)^2}$$

which is what we got earlier.

PINGBACKS

Pingback: Finite drop potential