INFINITE SQUARE WELL - CUBIC SINE INITIAL STATE

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.37.

As an example of the infinite square well potential suppose the particle starts off with the wave function

$$\Psi(x,0)=A\sin^{3}\left(\frac{\pi x}{a}\right) \tag{1}$$

for $0\le x\le a$ (and zero elsewhere).

First, we can find A by normalizing (using software for the integral):

$$\begin{aligned}
1 &= |A|^{2}\int_{0}^{a}\sin^{6}\left(\frac{\pi x}{a}\right)dx && (2)\\
&= |A|^{2}\frac{5}{16}a && (3)\\
A &= \frac{4}{\sqrt{5a}} && (4)
\end{aligned}$$

The general solution as a function of time is the series

$$\Psi(x,t)=\sum_{n=1}^{\infty}c_{n}\Psi_{n}(x,t) \tag{5}$$

where Ψ_n are the stationary states of the square well. We find the c_n by considering the sum at $t=0$:

$$\begin{aligned}
\Psi(x,0) &= \sum_{n=1}^{\infty}c_{n}\Psi_{n}(x,0) && (6)\\
&= \sqrt{\frac{2}{a}}\sum_{n=1}^{\infty}c_{n}\sin\left(\frac{n\pi x}{a}\right) && (7)
\end{aligned}$$

Because the stationary states are orthogonal functions, we have

$$c_n = \sqrt{\frac{2}{a}}\int_0^a \sin\left(\frac{n\pi x}{a}\right)\Psi(x,0) \tag{8}$$

$$= \sqrt{\frac{2}{a}}\frac{4}{\sqrt{5a}}\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin^3\left(\frac{\pi x}{a}\right)dx \tag{9}$$

$$= \frac{24}{5\pi}\frac{\sqrt{10}\sin(n\pi)}{n^4-10n^2+9} \tag{10}$$

A quick glance at this result might make you think that $c_n = 0$ for all n because of the sine term. However, we need to be careful, since the denominator factors to

$$n^4 - 10n^2 + 9 = (n-1)(n+1)(n-3)(n+3) \tag{11}$$

and thus has zeroes at $n = 1, 3$. We can redo these integrals for these specific values of n and we get

$$c_1 = \sqrt{\frac{2}{a}}\frac{4}{\sqrt{5a}}\int_0^a \sin\left(\frac{\pi x}{a}\right)\sin^3\left(\frac{\pi x}{a}\right)dx \tag{12}$$

$$= \frac{3}{\sqrt{10}} \tag{13}$$

$$c_3 = \sqrt{\frac{2}{a}}\frac{4}{\sqrt{5a}}\int_0^a \sin\left(\frac{3\pi x}{a}\right)\sin^3\left(\frac{\pi x}{a}\right)dx \tag{14}$$

$$= -\frac{1}{\sqrt{10}} \tag{15}$$

The full solution is therefore

$$\Psi(x,t) = \frac{3}{\sqrt{5a}}\sin\left(\frac{\pi x}{a}\right)e^{-i\pi^2\hbar t/2ma^2} - \frac{1}{\sqrt{5a}}\sin\left(\frac{3\pi x}{a}\right)e^{-9i\pi^2\hbar t/2ma^2} \tag{16}$$

The particle will be found with energy E_1 with probability of $|c_1|^2 = 0.9$ and with energy E_3 with probability 0.1. Thus

$$\langle E\rangle = \left(0.9 + 0.1\times 3^2\right)\frac{\pi^2\hbar^2}{2ma^2} \tag{17}$$

$$= 0.9\frac{\pi^2\hbar^2}{ma^2} \tag{18}$$

The mean position is found from

$$\langle x\rangle = \int_0^a x\left|\Psi(x,t)\right|^2 dx \tag{19}$$

Working out the integrand, we get

$$\left|\Psi(x,t)\right|^2 = -\frac{6}{5a}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{3\pi x}{a}\right)\left(\sin\left(\frac{E_1 t}{\hbar}\right)\sin\left(\frac{E_3 t}{\hbar}\right)+\cos\left(\frac{E_1 t}{\hbar}\right)\cos\left(\frac{E_3 t}{\hbar}\right)\right) \tag{20}$$

$$+\frac{2}{a}-\frac{9}{5a}\cos^2\left(\frac{\pi x}{a}\right)-\frac{1}{5a}\cos^2\left(\frac{3\pi x}{a}\right) \tag{21}$$

$$= -\frac{6}{5a}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{3\pi x}{a}\right)\cos\left(\frac{(E_3-E_1)\,t}{\hbar}\right)+\frac{2}{a}-\frac{9}{5a}\cos^2\left(\frac{\pi x}{a}\right)-\frac{1}{5a}\cos^2\left(\frac{3\pi x}{a}\right) \tag{22}$$

We can now do the integral using software with the result

$$\int_0^a x\left|\Psi(x,t)\right|^2 dx = \frac{a}{2} \tag{23}$$

The mean position is the midpoint of the well.