

INFINITE SQUARE WELL - CUBIC SINE INITIAL STATE

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.37.

As an example of the infinite square well potential suppose the particle starts off with the wave function

$$(0.1) \quad \Psi(x, 0) = A \sin^3\left(\frac{\pi x}{a}\right)$$

for $0 \leq x \leq a$ (and zero elsewhere).

First, we can find A by normalizing (using software for the integral):

$$(0.2) \quad 1 = |A|^2 \int_0^a \sin^6\left(\frac{\pi x}{a}\right) dx$$

$$(0.3) \quad = |A|^2 \frac{5}{16} a$$

$$(0.4) \quad A = \frac{4}{\sqrt{5a}}$$

The general solution as a function of time is the series

$$(0.5) \quad \Psi(x, t) = \sum_{n=1}^{\infty} c_n \Psi_n(x, t)$$

where Ψ_n are the stationary states of the square well. We find the c_n by considering the sum at $t = 0$:

$$(0.6) \quad \Psi(x, 0) = \sum_{n=1}^{\infty} c_n \Psi_n(x, 0)$$

$$(0.7) \quad = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right)$$

Because the stationary states are orthogonal functions, we have

$$(0.8) \quad c_n = \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi x}{a}\right) \Psi(x, 0) dx$$

$$(0.9) \quad = \sqrt{\frac{2}{a}} \frac{4}{\sqrt{5a}} \int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin^3\left(\frac{\pi x}{a}\right) dx$$

$$(0.10) \quad = \frac{24}{5\pi} \frac{\sqrt{10} \sin(n\pi)}{n^4 - 10n^2 + 9}$$

A quick glance at this result might make you think that $c_n = 0$ for all n because of the sine term. However, we need to be careful, since the denominator factors to

$$(0.11) \quad n^4 - 10n^2 + 9 = (n-1)(n+1)(n-3)(n+3)$$

and thus has zeroes at $n = 1, 3$. We can redo these integrals for these specific values of n and we get

$$(0.12) \quad c_1 = \sqrt{\frac{2}{a}} \frac{4}{\sqrt{5a}} \int_0^a \sin\left(\frac{\pi x}{a}\right) \sin^3\left(\frac{\pi x}{a}\right) dx$$

$$(0.13) \quad = \frac{3}{\sqrt{10}}$$

$$(0.14) \quad c_3 = \sqrt{\frac{2}{a}} \frac{4}{\sqrt{5a}} \int_0^a \sin\left(\frac{3\pi x}{a}\right) \sin^3\left(\frac{\pi x}{a}\right) dx$$

$$(0.15) \quad = -\frac{1}{\sqrt{10}}$$

The full solution is therefore

$$(0.16) \quad \Psi(x, t) = \frac{3}{\sqrt{5a}} \sin\left(\frac{\pi x}{a}\right) e^{-i\pi^2 \hbar t / 2ma^2} - \frac{1}{\sqrt{5a}} \sin\left(\frac{3\pi x}{a}\right) e^{-9i\pi^2 \hbar t / 2ma^2}$$

The particle will be found with energy E_1 with probability of $|c_1|^2 = 0.9$ and with energy E_3 with probability 0.1. Thus

$$(0.17) \quad \langle E \rangle = (0.9 + 0.1 \times 3^2) \frac{\pi^2 \hbar^2}{2ma^2}$$

$$(0.18) \quad = 0.9 \frac{\pi^2 \hbar^2}{ma^2}$$

The mean position is found from

$$(0.19) \quad \langle x \rangle = \int_0^a x |\Psi(x,t)|^2 dx$$

Working out the integrand, we get

$$(0.20) \quad |\Psi(x,t)|^2 = -\frac{6}{5a} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{3\pi x}{a}\right) \left(\sin\left(\frac{E_1 t}{\hbar}\right) \sin\left(\frac{E_3 t}{\hbar}\right) + \cos\left(\frac{E_1 t}{\hbar}\right) \cos\left(\frac{E_3 t}{\hbar}\right) \right)$$

$$(0.21) \quad + \frac{2}{a} - \frac{9}{5a} \cos^2\left(\frac{\pi x}{a}\right) - \frac{1}{5a} \cos^2\left(\frac{3\pi x}{a}\right)$$

$$(0.22) \quad = -\frac{6}{5a} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{3\pi x}{a}\right) \cos\left(\frac{(E_3 - E_1)t}{\hbar}\right) + \frac{2}{a} - \frac{9}{5a} \cos^2\left(\frac{\pi x}{a}\right) - \frac{1}{5a} \cos^2\left(\frac{3\pi x}{a}\right)$$

We can now do the integral using software with the result

$$(0.23) \quad \int_0^a x |\Psi(x,t)|^2 dx = \frac{a}{2}$$

The mean position is the midpoint of the well.