HARMONIC OSCILLATOR - EXAMPLE STARTING STATE

Suppose a particle in the harmonic oscillator potential starts out in the state

$$\Psi(x,0) = A \left( 1 - 2 \sqrt{\frac{m\omega}{\hbar}} \right)^2 e^{-m\omega x^2/2\hbar} \quad (1)$$

We can find the expectation value of the energy by expressing the given wave function as a linear combination of Hermite polynomials, since these form the orthonormal basis of solutions in the harmonic oscillator potential. Using the definitions

$$\alpha \equiv \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} \quad (2)$$

$$\xi \equiv \sqrt{\frac{m\omega}{\hbar}} x \quad (3)$$

we have

$$\Psi(x,0) = A(1-2\xi)^2 e^{-\xi^2/2} \quad (4)$$

Normalizing $\Psi(x,0)$, by hand or using Maple, we find

$$\int_{-\infty}^{\infty} |\Psi(\xi,0)|^2 \, d\xi = 1 \quad (5)$$

$$|A|^2 = \frac{\alpha^2}{25} \quad (6)$$

The stationary states of the Harmonic oscillator are

$$\psi_n(\xi) = \alpha \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2} \quad (7)$$
Taking $A$ to be real and positive, and expanding in terms of the Hermite polynomials, we have

$$\Psi(x,0) = \frac{\alpha}{5}(4\xi^2 - 4\xi + 1)e^{-\xi^2/2}$$  

$$= \frac{\alpha}{5}(H_2 - 2H_1 + 3H_0)e^{-\xi^2/2}$$  

$$= \frac{1}{5}(2\sqrt{2}\psi_2 - 2\sqrt{2}\psi_1 + 3\psi_0)$$

The expectation value for the energy can now be calculated from the energies of the lowest 3 states of the harmonic oscillator: $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$. We get:

$$\langle E \rangle = \frac{\hbar\omega}{25} \left(8 \times \frac{5}{2} + 8 \times \frac{3}{2} + 9 \times \frac{1}{2}\right) = \frac{73}{50}\hbar\omega$$

Now suppose that at some later time $T$ the wave function has changed to

$$\Psi(x,T) = B \left(1 + 2\sqrt{\frac{m\omega}{\hbar}}x\right)^2 e^{-m\omega x^2/2\hbar}$$  

$$\Psi(\xi,T) = B \left(1 + 2\xi\right)^2 e^{-\xi^2/2}$$

(The only changes from $\Psi(x,0)$ are the plus sign in the middle factor and the different normalization constant $B$.)

By normalizing the new wave function, we find that $|B|^2 = \frac{\alpha^2}{25} = |A|^2$. This time, we can’t assume that $B$ is real, but it can differ from $A$ by a complex factor $q$ such that $|q|^2 = 1$. If we convert the given wave function into Hermite polynomials as in part (a), we get:

$$\Psi(\xi,T) = \frac{\alpha}{5}(H_2 + 2H_1 + 3H_0)e^{-\xi^2/2}$$

$$= \frac{q}{5}(2\sqrt{2}\psi_2 + 2\sqrt{2}\psi_1 + 3\psi_0)$$

Thus the only difference between this function and the initial function (apart from the factor $q$) is that the coefficient of $\psi_1$ has changed sign. Since the general time-dependent solution is given by (with the lower limit on the sum changed to 0 because the ground state for the harmonic oscillator has index 0 rather than 1):

$$\Psi(x,t) = \sum_{n=0}^{\infty} c_n \psi_n(x)e^{-iE_n t/\hbar}$$
we need to find the time $T$ such that $-e^{-iE_1 T/\hbar} = e^{-iE_0 T/\hbar} = e^{-iE_2 T/\hbar}$.

Substituting the energies, we find:

$$-e^{-3i\omega T/2} = e^{-i\omega T/2} = e^{-5i\omega T/2}$$  \hspace{1cm} (17)

That is

$$\frac{3\omega T}{2} = \frac{\omega T}{2} + (2m + 1)\pi$$  \hspace{1cm} (18)

$$\frac{5\omega T}{2} = \frac{\omega T}{2} + 2r\pi$$  \hspace{1cm} (19)

for some integers $m$ and $r$. From the first equation, we get $\omega T = (2m + 1)\pi$ and from the second, $2\omega T = 2r\pi$. The smallest $T > 0$ occurs when $m = 0$ and $r = 1$, giving $T = \pi/\omega$. Then $e^{-3i\pi/2} = +i$, and $e^{-5i\pi/2} = e^{-5\pi i/2} = -i$.

By setting $q = -i$, we get the wave function at $t = T = \pi/\omega$:

$$\Psi(x, T) = -\frac{i}{5}(2\sqrt{2}\psi_2 + 2\sqrt{2}\psi_1 + 3\psi_0)$$  \hspace{1cm} (20)