

## HARMONIC OSCILLATOR - EXAMPLE STARTING STATE

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.41.

Suppose a particle in the harmonic oscillator potential starts out in the state

$$(1) \quad \Psi(x, 0) = A \left( 1 - 2\sqrt{\frac{m\omega}{\hbar}}x \right)^2 e^{-m\omega x^2/2\hbar}$$

We can find the expectation value of the energy by expressing the given wave function as a linear combination of Hermite polynomials, since these form the orthonormal basis of solutions in the harmonic oscillator potential. Using the definitions

$$(2) \quad \alpha \equiv \left( \frac{m\omega}{\pi\hbar} \right)^{1/4}$$

$$(3) \quad \xi \equiv \sqrt{\frac{m\omega}{\hbar}}x$$

we have

$$(4) \quad \Psi(x, 0) = A(1 - 2\xi)^2 e^{-\xi^2/2}$$

Normalizing  $\Psi(x, 0)$ , by hand or using Maple, we find

$$(5) \quad \int_{-\infty}^{\infty} |\Psi(\xi, 0)|^2 d\xi = 1$$

$$(6) \quad |A|^2 = \frac{\alpha^2}{25}$$

The stationary states of the Harmonic oscillator are

$$(7) \quad \psi_n(\xi) = \alpha \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}$$

Taking  $A$  to be real and positive, and expanding in terms of the Hermite polynomials, we have

$$(8) \quad \Psi(x, 0) = \frac{\alpha}{5}(4\xi^2 - 4\xi + 1)e^{-\xi^2/2}$$

$$(9) \quad = \frac{\alpha}{5}(H_2 - 2H_1 + 3H_0)e^{-\xi^2/2}$$

$$(10) \quad = \frac{1}{5}(2\sqrt{2}\psi_2 - 2\sqrt{2}\psi_1 + 3\psi_0)$$

The expectation value for the energy can now be calculated from the energies of the lowest 3 states of the harmonic oscillator:  $E_n = (n + \frac{1}{2})\hbar\omega$ . We get:

$$(11) \quad \langle E \rangle = \frac{\hbar\omega}{25} \left( 8 \times \frac{5}{2} + 8 \times \frac{3}{2} + 9 \times \frac{1}{2} \right) = \frac{73}{50}\hbar\omega$$

Now suppose that at some later time  $T$  the wave function has changed to

$$(12) \quad \Psi(x, T) = B \left( 1 + 2\sqrt{\frac{m\omega}{\hbar}}x \right)^2 e^{-m\omega x^2/2\hbar}$$

$$(13) \quad \Psi(\xi, T) = B(1 + 2\xi)^2 e^{-\xi^2/2}$$

(The only changes from  $\Psi(x, 0)$  are the plus sign in the middle factor and the different normalization constant  $B$ .)

By normalizing the new wave function, we find that  $|B|^2 = \frac{\alpha^2}{25} = |A|^2$ . This time, we can't assume that  $B$  is real, but it can differ from  $A$  by a complex factor  $q$  such that  $|q|^2 = 1$ . If we convert the given wave function into Hermite polynomials as in part (a), we get:

$$(14) \quad \Psi(\xi, T) = q \frac{\alpha}{5}(H_2 + 2H_1 + 3H_0)e^{-\xi^2/2}$$

$$(15) \quad = \frac{q}{5}(2\sqrt{2}\psi_2 + 2\sqrt{2}\psi_1 + 3\psi_0)$$

Thus the only difference between this function and the initial function (apart from the factor  $q$ ) is that the coefficient of  $\psi_1$  has changed sign. Since the general time-dependent solution is given by (with the lower limit on the sum changed to 0 because the ground state for the harmonic oscillator has index 0 rather than 1):

$$(16) \quad \Psi(x, t) = \sum_{n=0}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar}$$

we need to find the time  $T$  such that  $-e^{-iE_1 T/\hbar} = e^{-iE_0 T/\hbar} = e^{-iE_2 T/\hbar}$ . Substituting the energies, we find:

$$(17) \quad -e^{-3i\omega T/2} = e^{-i\omega T/2} = e^{-5i\omega T/2}$$

That is

$$(18) \quad \frac{3\omega T}{2} = \frac{\omega T}{2} + (2m+1)\pi$$

$$(19) \quad \frac{5\omega T}{2} = \frac{\omega T}{2} + 2r\pi$$

for some integers  $m$  and  $r$ . From the first equation, we get  $\omega T = (2m+1)\pi$  and from the second,  $2\omega T = 2r\pi$ . The smallest  $T > 0$  occurs when  $m = 0$  and  $r = 1$ , giving  $T = \pi/\omega$ . Then  $e^{-3i\pi/2} = +i$ , and  $e^{-\pi i/2} = e^{-5\pi i/2} = -i$ . By setting  $q = -i$ , we get the wave function at  $t = T = \pi/\omega$ :

$$(20) \quad \Psi(x, T) = -\frac{i}{5}(2\sqrt{2}\psi_2 + 2\sqrt{2}\psi_1 + 3\psi_0)$$