HARMONIC OSCILLATOR - EXAMPLE STARTING STATE

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.41.

Suppose a particle in the harmonic oscillator potential starts out in the state

$$\Psi(x,0)=A\left(1-2\sqrt{\frac{m\omega}{\hbar}}x\right)^{2}e^{-m\omega x^{2}/2\hbar} \tag{0.1}$$

We can find the expectation value of the energy by expressing the given wave function as a linear combination of Hermite polynomials, since these form the orthonormal basis of solutions in the harmonic oscillator potential. Using the definitions

$$\alpha \equiv \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \tag{0.2}$$

$$\xi \equiv \sqrt{\frac{m\omega}{\hbar}}x \tag{0.3}$$

we have

$$\Psi(x,0)=A(1-2\xi)^{2}e^{-\xi^{2}/2} \tag{0.4}$$

Normalizing $\Psi(x,0)$, by hand or using Maple, we find

$$\int_{-\infty}^{\infty}|\Psi(\xi,0)|^{2}d\xi = 1 \tag{0.5}$$

$$|A|^{2} = \frac{\alpha^{2}}{25} \tag{0.6}$$

The stationary states of the Harmonic oscillator are

$$\psi_{n}(\xi)=\alpha\frac{1}{\sqrt{2^{n}n!}}H_{n}(\xi)e^{-\xi^{2}/2} \tag{0.7}$$

Taking A to be real and positive, and expanding in terms of the Hermite polynomials, we have

$$\Psi(x,0) = \frac{\alpha}{5}(4\xi^2 - 4\xi + 1)e^{-\xi^2/2} \tag{0.8}$$

$$= \frac{\alpha}{5}(H_2 - 2H_1 + 3H_0)e^{-\xi^2/2} \tag{0.9}$$

$$= \frac{1}{5}(2\sqrt{2}\psi_2 - 2\sqrt{2}\psi_1 + 3\psi_0) \tag{0.10}$$

The expectation value for the energy can now be calculated from the energies of the lowest 3 states of the harmonic oscillator: $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$. We get:

$$\langle E\rangle = \frac{\hbar\omega}{25}\left(8\times\frac{5}{2} + 8\times\frac{3}{2} + 9\times\frac{1}{2}\right) = \frac{73}{50}\hbar\omega \tag{0.11}$$

Now suppose that at some later time T the wave function has changed to

$$\Psi(x,T) = B\left(1 + 2\sqrt{\frac{m\omega}{\hbar}}x\right)^2 e^{-m\omega x^2/2\hbar} \tag{0.12}$$

$$\Psi(\xi,T) = B(1+2\xi)^2 e^{-\xi^2/2} \tag{0.13}$$

(The only changes from $\Psi(x,0)$ are the plus sign in the middle factor and the different normalization constant B.)

By normalizing the new wave function, we find that $|B|^2 = \frac{\alpha^2}{25} = |A|^2$. This time, we can't assume that B is real, but it can differ from A by a complex factor q such that $|q|^2 = 1$. If we convert the given wave function into Hermite polynomials as in part (a), we get:

$$\Psi(\xi,T) = q\frac{\alpha}{5}(H_2 + 2H_1 + 3H_0)e^{-\xi^2/2} \tag{0.14}$$

$$= \frac{q}{5}(2\sqrt{2}\psi_2 + 2\sqrt{2}\psi_1 + 3\psi_0) \tag{0.15}$$

Thus the only difference between this function and the initial function (apart from the factor q) is that the coefficient of ψ_1 has changed sign. Since the general time-dependent solution is given by (with the lower limit on the sum changed to 0 because the ground state for the harmonic oscillator has index 0 rather than 1):

$$\Psi(x,t)=\sum_{n=0}^{\infty}c_n\psi_n(x)e^{-iE_nt/\hbar} \tag{0.16}$$

we need to find the time T such that $-e^{-iE_1T/\hbar}=e^{-iE_0T/\hbar}=e^{-iE_2T/\hbar}$. Substituting the energies, we find:

$$-e^{-3i\omega T/2}=e^{-i\omega T/2}=e^{-5i\omega T/2} \tag{0.17}$$

That is

$$\frac{3\omega T}{2} = \frac{\omega T}{2}+(2m+1)\pi \tag{0.18}$$

$$\frac{5\omega T}{2} = \frac{\omega T}{2}+2r\pi \tag{0.19}$$

for some integers m and r. From the first equation, we get $\omega T=(2m+1)\pi$ and from the second, $2\omega T=2r\pi$. The smallest $T>0$ occurs when $m=0$ and $r=1$, giving $T=\pi/\omega$. Then $e^{-3i\pi/2}=+i$, and $e^{-\pi i/2}=e^{-5\pi i/2}=-i$. By setting $q=-i$, we get the wave function at $t=T=\pi/\omega$:

$$\Psi(x,T)=-\frac{i}{5}(2\sqrt{2}\psi_2+2\sqrt{2}\psi_1+3\psi_0) \tag{0.20}$$