

FREE PARTICLE - TRAVELLING WAVE PACKET

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.43.

We've looked at the stationary Gaussian wave packet for the free particle. The initial wave function in that case was

$$\Psi(x, 0) = Ae^{-ax^2} \quad (1)$$

We can turn this into a travelling Gaussian wave by adding a factor to the wave function:

$$\Psi(x, 0) = Ae^{-ax^2} e^{ilx} \quad (2)$$

where l is a real constant.

Since we have added only a complex exponential, the normalization condition is the same as for the stationary case:

$$A = \left(\frac{2a}{\pi}\right)^{1/4} \quad (3)$$

To find $\Psi(x, t)$ we follow the same procedure as in the stationary case. So we get

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k) e^{ikx} e^{-i\hbar k^2 t/2m} dk \quad (4)$$

Given the initial wave function, we can find $\phi(k)$ via Plancherel's theorem:

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(x, 0) e^{-ikx} dx \quad (5)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{1/4} \int_{-\infty}^{\infty} e^{-ax^2 - ikx + ilx} dx \quad (6)$$

$$= \left(\frac{1}{2\pi a}\right)^{1/4} e^{-(k-l)^2/4a} \quad (7)$$

So we can now find the general solution:

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \hbar k^2 t / 2m)} dk \quad (8)$$

$$= \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{(-ax^2 + i(lx - \hbar l^2 t / 2m)) / (1 + 2i\hbar at / m)}}{\sqrt{1 + 2i\hbar at / m}} \quad (9)$$

where Maple was used for the integral.

Calculating $|\Psi(x, t)|^2$ can be done using Maple, with the result:

$$|\Psi(x, t)|^2 = \sqrt{\frac{2}{\pi}} w e^{-2w^2(\hbar l t / m - x)^2} \quad (10)$$

with

$$w \equiv \left(\frac{a}{1 + (2\hbar at / m)^2}\right)^{1/2} \quad (11)$$

The results above reduce to the stationary wave packet when $l = 0$.

At $t = 0$, $w = \sqrt{a}$, so $|\Psi(x, 0)|^2 = \sqrt{2a/\pi} e^{-2ax^2}$ which is correct. The wave packet at $t = 0$ is therefore a Gaussian centred at $x = 0$. As t increases, w gets smaller but in this case, the peak of the Gaussian moves according to $x_{peak} = \hbar l t / m$. The speed of the peak is $x/t = \hbar l / m$.

By direct integration we find, $\langle x \rangle = \hbar l t / m$. Calculating the other means requires a bit of effort but we can use Maple to do most of it. The results are:

$$\langle p \rangle = l\hbar \quad (12)$$

$$\langle x^2 \rangle = \frac{1 + (2a\hbar t / m)^2 + a(2\hbar l t / m)^2}{4a} \quad (13)$$

$$\langle p^2 \rangle = \hbar^2(a + l^2) \quad (14)$$

All these results reduce to those for the stationary wave packet from problem 2.22 when $l = 0$.

The uncertainty principle thus becomes

$$\sigma_x \sigma_p = \sqrt{(\langle x^2 \rangle - \langle x \rangle^2)(\langle p^2 \rangle - \langle p \rangle^2)} \quad (15)$$

$$= \frac{\hbar}{2} \sqrt{1 + (2\hbar at / m)^2} \quad (16)$$

which is the same result as in the stationary wave packet. Thus although the

packet here travels with a constant speed, it spreads out at the same rate as the stationary packet.

PINGBACKS

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