

FREE PARTICLE - TRAVELLING WAVE PACKET

Link to: [physicspages home page](#).

To leave a comment or report an error, please use the [auxiliary blog](#).

References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.43.

We've looked at the stationary Gaussian wave packet for the free particle. The initial wave function in that case was

$$(1) \quad \Psi(x, 0) = Ae^{-ax^2}$$

We can turn this into a travelling Gaussian wave by adding a factor to the wave function:

$$(2) \quad \Psi(x, 0) = Ae^{-ax^2} e^{ilx}$$

where l is a real constant.

Since we have added only a complex exponential, the normalization condition is the same as for the stationary case:

$$(3) \quad A = \left(\frac{2a}{\pi} \right)^{1/4}$$

To find $\Psi(x, t)$ we follow the same procedure as in the stationary case. So we get

$$(4) \quad \Psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k) e^{ikx} e^{-i\hbar k^2 t / 2m} dk$$

Given the initial wave function, we can find $\phi(k)$ via Plancherel's theorem:

$$(5) \quad \phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(x,0) e^{-ikx} dx$$

$$(6) \quad = \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{1/4} \int_{-\infty}^{\infty} e^{-ax^2 - ikx + ilx} dx$$

$$(7) \quad = \left(\frac{1}{2\pi a}\right)^{1/4} e^{-(k-l)^2/4a}$$

So we can now find the general solution:

$$(8) \quad \Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \hbar k^2 t/2m)} dk$$

$$(9) \quad = \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{(-ax^2 + i(lx - \hbar l^2 t/2m))/(1 + 2i\hbar at/m)}}{\sqrt{1 + 2i\hbar at/m}}$$

where Maple was used for the integral.

Calculating $|\Psi(x,t)|^2$ can be done using Maple, with the result:

$$(10) \quad |\Psi(x,t)|^2 = \sqrt{\frac{2}{\pi}} w e^{-2w^2(\hbar l t/m - x)^2}$$

with

$$(11) \quad w \equiv \left(\frac{a}{1 + (2\hbar at/m)^2}\right)^{1/2}$$

The results above reduce to the stationary wave packet when $l = 0$.

At $t = 0$, $w = \sqrt{a}$, so $|\Psi(x,0)|^2 = \sqrt{2a/\pi} e^{-2ax^2}$ which is correct. The wave packet at $t = 0$ is therefore a Gaussian centred at $x = 0$. As t increases, w gets smaller but in this case, the peak of the Gaussian moves according to $x_{peak} = \hbar l t/m$. The speed of the peak is $x/t = \hbar l/m$.

By direct integration we find, $\langle x \rangle = \hbar l t/m$. Calculating the other means requires a bit of effort but we can use Maple to do most of it. The results are:

$$(12) \quad \langle p \rangle = l\hbar$$

$$(13) \quad \langle x^2 \rangle = \frac{1 + (2a\hbar t/m)^2 + a(2\hbar l t/m)^2}{4a}$$

$$(14) \quad \langle p^2 \rangle = \hbar^2(a + l^2)$$

All these results reduce to those for the stationary wave packet from problem 2.22 when $l = 0$.

The uncertainty principle thus becomes

$$(15) \quad \sigma_x \sigma_p = \sqrt{(\langle x^2 \rangle - \langle x \rangle^2)(\langle p^2 \rangle - \langle p \rangle^2)}$$

$$(16) \quad = \frac{\hbar}{2} \sqrt{1 + (2\hbar a t / m)^2}$$

which is the same result as in the stationary wave packet. Thus although the packet here travels with a constant speed, it spreads out at the same rate as the stationary packet.

PINGBACKS

Pingback: Energy-time uncertainty principle: Gaussian free particle

Pingback: Free particle in momentum space